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Explorations in the theory of the multiproduct firm

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Explorations in the theory of the multiproduct firm

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CHAPTER I. INTRODUCTION

The objective of this dissertation is to delve a bit deeper than has been done before into various aspects of the theory of the multiproduct firm. In particular, the work is of the nature of an exploration into a quite complex[1] area and as such no claim is made with respect to having presented a complete, overall, theory of the multiproduct firm.

The presentation itself is divided into three chapters and an Appendix. Chapter II, then, deals with the short run cost minimization problem of a multiproduct firm operating in a productive atmosphere characterized by independent production. That is to say the level of output of any one final good is not affected directly by the output level of any other final good the firm produces. Mathematically such a situation amounts to the positing of r independent production functions, each one pertaining to one of the r goods the firm produces.

In particular the formulation of the problem is such that an initial allocation of fixed factors is made over the various r production processes of interest. This initial allocation is made directly previous to the particular short run of interest and it is assumed that no reallocation of those fixed factors is possible for the duration of the period.

In this problem, as well as those treated in the following chapters, the fixed factors themselves are given a somewhat more explicit role

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[1]Perhaps one should add "relatively neglected."

than is usually the case in contemporary neo-classical production theory. In particular, the fixed factors appear in production functions at usage rather than capacity levels, a consideration which allows one to take explicit account of excess capacity in a multiproduct (or single product) firm that uses both fixed and variable factors of production.

The third chapter again deals with cost minimization but the treatment is somewhat briefer than the previous problem in that it does not deal with the vagaries of excess capacity to the extent that Chapter II does. The production function used to characterize the productive atmosphere in which the firm under consideration finds itself is almost identical to that denoted by Dano [8] as representative of what he calls "classical joint production" and which in turn is characterized by his mentor, the Nobel Laureate, Frisch [11], as possessing a degree of assortment one less than the number of goods being produced by the firm.

Again, as in the previous problem, the explicit usage levels of fixed as well as variable input levels are included in the production function. However, in contrast to that previous problem, here fixed factors may be thought of as amenable to being, costlessly, switched from production of one good to that of another good. This last consideration leads to the positing of an opportunity cost associated with the assignment of scarce factors. The existence of such a cost is of particular interest in Chapter IV where the firm is assumed to produce under these conditions given in Chapter III.

Profit maximization is the concern of the fourth and final chapter. Here the concepts of nonprice offer variation[1] and the sales function of the firm are introduced, discussed, and incorporated into the analysis. In particular, prices and N.P.O.V.'s become, as opposed to quantities, the relevant decision or control variables of the firm in its attempt to maximize net revenue.

The results of this last chapter highlight, rather clearly, the supposition that the multiproduct firm is concerned with the effects that changes in decision variables induce in sales and production throughout its posited product line. This consideration is greatly stressed, not only through the very nature of the mathematical terms comprising the necessary conditions to be inspected but also through the economic interpretation of such conditions.

The first order necessary conditions are somewhat different than those which usually appear in contemporary textbook treatment of the firm in that opportunity costs associated with the allocation of scarce factors at the margin are treated on an equal footing with marginal variable costs. The rationale for such a treatment is revealed through a demonstration involving the second order necessary conditions for a local maximum in an appropriately defined problem.

Throughout, interest is focused almost exclusively[2] upon production devoted to current sales, that is, production of saleable products and

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[1]Henceforth to be referred to as N.P.O.V.

[2]The almost refers to a brief excursion in Chapter II.

N.P.O.V.'s destined to be sold and offered during the short run of interest. Therefore it is assumed that the firm of interest is operating with zero inventory accumulation, that is, as Holdren [15, p. 580] puts it, "We assume unless otherwise stated, that the firm is not undertaking intentional inventory accumulation or decumulation." This last assumption, often implicity made but rarely explicitly stated, allows one to set production equal to sales.

The mathematical technique employed throughout is that of nonlinear programming. This technique was chosen because it gives one the ability to consider both equality and inequality constraints in the same problem in a somewhat natural fashion; natural pertaining to the supposition that this technique seems to be a tailor-made language for the economist.[1]

The technique is used to postulate necessary conditions for a maximum (minimum) of a real valued twice differentiable function defined on some open subset of R^n . The necessary conditions postulated are of both the first and second order, the latter conditions pertaining to a local maximum (minimum) and the former pertaining to any maximum (minimum). The normality condition assumed, that is, the condition the fulfillment of which allows one to postulate that the aforementioned necessary conditions are indeed necessary, is the rank condition of the Arrow-Hurwicz-Uzawa [2] constraint qualification theorem. A general nonlinear programming problem incorporating mixed constraints is presented

[1] That nonlinear programming is such a language has been impressed upon me (quite often) by my major professor Bob Holdren.

as an appendix and should be of interest in that it is specifically directed towards presenting all the conditions incorporated in the development of the three chapters to follow.

CHAPTER II. COST MINIMIZATION UNDER CONDITIONS OF INDEPENDENT PRODUCTION

The work in this chapter is concerned with the cost minimization problem of a multiproduct firm producing r goods under conditions of independent production. Mathematically such a situation amounts to the positing of r independent production functions.

The problem is cast in terms of a short run situation, and in particular, of the many short runs one might consider, this one involves a production period such that at the outset of the period the fixed factors of production are allocated for use among the various production processes. Furthermore, once this initial allocation has been made there is no possibility of reallocation of those fixed factors; a consideration which holds for the duration of the period under consideration.

Previously, a similar problem was treated by Ralph Pfouts [20], however he allowed switching (reallocation) of fixed factors at any time during the production period if warranted by a compensation greater than the costs of the reallocation itself. Pfouts' treatment is repeated in Benavie [5] and Ferguson [9] and is adcpted by Naylor [19] in the context of a profit maximization problem and again, but to a lesser degree, by Swenson [25].

Some of the notation to be used in this, as well as the following chapters, and explanation thereof, is presented directly below:

$$V = (v_1, \ldots, v_j, \ldots, v_s), \quad v_j \geq 0, \quad j = (1, \ldots, s) ,$$

where v_j is the level of usage of the variable factor of production V_j.

$$\overline{Y} = (\overline{y}_1, \ldots, \overline{y}_k, \ldots, \overline{y}_t), \quad \overline{y}_k \geq 0, \quad k = (1, \ldots, t) ,$$

where $\overline{y}_k$ is the maximum amount available of the fixed factor of production Y_k.

$$Y = (y_1, \ldots, y_k, \ldots, y_t), \quad y_k \geq 0, \quad k = (1, \ldots, t) ,$$

where y_k is the level of usage of Y_k.

$$Q = (q_1, \ldots, q_i, \ldots, q_r), \quad q_i \geq 0, \quad i = (1, \ldots, r) ,$$

where q_i is the level of output of good Q_i.[1]

The production functions are given by

$$q_i = q_i(v_{i1}, \ldots, v_{ij}, \ldots, v_{is}, y_{i1}, \ldots, y_{ik}, \ldots, y_{it}),$$

$$i = (1, \ldots, r) \tag{2.1}$$

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[1]Some of the goods being produced are saleable output while others are N.P.O.V.'s. However, there is not need to differentiate between them until Chapter IV.

$v_{ij} \geq 0$; v_{ij} denotes the usage level of the j^{th} variable input used in the production of the i^{th} good. $y_{ik} \geq 0$; y_{ik} denotes the usage level of the k^{th} fixed input used in the production of the i^{th} good.

The cost function is given by

$$T.C. = C(v_1, \ldots, v_j, \ldots, v_s) + F \tag{2.2}$$

where

$$C = \sum_{i=1}^{r} C_i(v_{i1}, \ldots, v_{ij}, \ldots, v_{is}) ,$$

and F represents fixed costs.

Finally, the initial allocation of fixed factors is represented by

$$\bar{y}_k = \sum_{i=1}^{r} \bar{y}_{ik}, \quad k = (1, \ldots, t) \tag{2.3}$$

Mathematically, the cost minimization problem is as follows:

Minimize

$$C = \sum_{i=1}^{r} C_i(v_{i1}, \ldots, v_{ij}, \ldots, v_{is}) + F \tag{2.4}$$

subject to

$$\bar{q}_1 - q_1(v_{11}, \ldots, v_{ij}, \ldots, v_{1s}, y_{11}, \ldots, y_{1k}, \ldots, y_{1t}) = 0$$

$$\bar{q}_i - q_i(v_{i1}, \ldots, v_{ij}, \ldots, v_{is}, y_{i1}, \ldots, y_{ik}, \ldots, y_{it}) = 0$$

$$\bar{q}_r - q_r(v_{r1}, \ldots, v_{rj}, \ldots, v_{rs}, y_{r1}, \ldots, y_{rk}, \ldots, y_{rt}) = 0$$

$$\bar{y}_{ik} - y_{ik} \geq 0, \quad i = (1, \ldots, r), \quad k = (1, \ldots, t)$$

$$v_{ij} \geq 0, \quad j = (1, \ldots, s)$$

$$y_{ik} \geq 0 .$$ [1]

Assume now that (V^*, Y^*) is a feasible point of the constraint set and that at (V^*, Y^*) the first $\ell < s + t - r$ of the inequality constraints are binding. Consider now the jacobian matrix [J] (below) evaluated at (V^*, Y^*),

[1]All functions and constraints are assumed to be real valued and to have the necessary differentiability properties on X, an open subset of R^{s+t}.

$$
\underset{(\ell+r)x(s+t)}{(J)} = \begin{bmatrix}
\frac{\partial q_1}{\partial v_1} \cdots \frac{\partial q_1}{\partial v_j} \cdots \frac{\partial q_1}{\partial y_1} \frac{\partial q_1}{\partial y_1} \cdots \frac{\partial q_1}{\partial y_k} \cdots \frac{\partial q_1}{\partial y_t} \\
\\
\frac{\partial q_i}{\partial v_1} \cdots \frac{\partial q_i}{\partial v_j} \cdots \frac{\partial q_i}{\partial v_s} \frac{\partial q_i}{\partial y_1} \cdots \frac{\partial q_i}{\partial y_k} \cdots \frac{\partial q_i}{\partial y_t} \\
\\
\frac{\partial q_r}{\partial v_1} \cdots \frac{\partial q_r}{\partial v_j} \cdots \frac{\partial q_r}{\partial v_s} \frac{\partial q_r}{\partial y_1} \cdots \frac{\partial q_r}{\partial y_k} \cdots \frac{\partial q_r}{\partial y_t} \\
\\
0 \\
\\
\qquad 0 \qquad\qquad\qquad A \\
\\
0
\end{bmatrix}^{1} \qquad (2.5)
$$

$$(\ell+r) \text{ x } (s+t)$$

If the rank of (J) is ℓ+r, where $\ell+r < s+t$ then one can say that (V^*, Y^*) is a regular point in the sense that it satisfies a constraint qualification or normality condition. The above criterion for normality is developed by Arrow, Hurwicz and Uzawa [2] and is particularly useful in the case in which both inequality and equality constraints must be considered and one does not wish to start out by positing restrictive

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[1] A is a portion of the jacobian of an arbitrary set of ℓ inequality constraints.

assumptions on the curvature of the constraints.[1] Economically one might say that the assumption of normality rules out degenerate cases; cases in which there is no choice to be made between alternate production points and therefore no minimization problem to be considered.[2] In particular the satisfaction of such a condition is sufficient to proceed to the inspection of first order conditions emanating from the lagrangean in the sense that such conditions are indeed necessary for a minimum at (V^*, Y^*).

The lagrangean expression is written as follows

$$L(V,Y,\lambda,U) = \sum_{i=1}^{r} C_i(v_{i1}, \ldots, v_{ij}, \ldots, v_{is}) + F$$

$$+ \sum_{i=1}^{r} \lambda_i[\bar{q}_i - q_i(v_{i1}, \ldots, v_{ij}, \ldots, v_{is}, y_{i1}, \ldots, y_{ik}, \ldots, y_{it})]$$

$$+ \sum_{i=1}^{r} \sum_{k=1}^{t} U_{ik}(\bar{y}_{ik} - y_{ik}) \qquad (2.6)$$

where

[1]Especially true in this case since production functions comprise the r equality constraints. Also note that the positing of the subset of inequality constraints which are binding at (V^*, Y^*) is any arbitrary choice.

[2]This is of course an oversimplified explanation in a mathematical sense but is nonetheless the economically relevant one.

In order for the point (V^*, Y^*) to minimize C subject to the posited constraints the following conditions must be satisfied when evaluated at $(V^*, Y^*, \lambda^*, U^*)$,

$$\frac{\partial C_i}{\partial V_{i1}} - \lambda i(\frac{\partial q_i}{\partial V_{ij}}) \geq 0$$

$$\frac{\partial C_i}{\partial V_{ij}} - \lambda i(\frac{\partial q_i}{\partial V_{ij}}) \geq 0$$

$$\frac{\partial C_i}{\partial V_{is}} - \lambda_i(\frac{\partial q_i}{\partial V_{is}} \geq 0 \qquad (2.7)$$

where

$i = (1, \ldots, r), \quad j = (1, \ldots, s)$

$$-\lambda_i(\frac{\partial q_i}{\partial y_{i1}}) - U_{i1} \geq 0$$

$$-\lambda_i(\frac{\partial q_i}{\partial y_{ij}}) - U_{ij} \geq 0$$

$$-\lambda_i(\frac{\partial q_i}{\partial y_{it}}) - U_{it} \geq 0 \qquad (2.8)$$

where

$i = (1, \ldots, r)$

and

$$U^*_{ik} \geq 0, \quad V^*_{ij} \geq 0, \quad y^*_{ik} \geq 0 \text{ and } V^*, Y^* \varepsilon K$$

where K is the feasible set.

The conditions denoted by (2.7) are comprised of r sets of inequalities where each set itself is comprised of s inequalities. Assuming that all variable inputs are used at positive levels,[1] and dividing the j^{th} by the k^{th} equation of the i^{th} set of equations denoted by (2.7), where $k, j \varepsilon (1, \ldots, s)$, $i \varepsilon (1, \ldots, r)$, yields the following conditions

$$\frac{\partial C_i/\partial v_{ij}}{\partial C_i/\partial v_{ik}} = \frac{\partial q_i/\partial v_{ij}}{\partial q_i/\partial v_{ik}} \quad i = (1, \ldots, r),\ j,\ k\varepsilon(1, \ldots, s)\ . \tag{2.9}$$

Assuming that the firm buys its variable factors of production in perfectly competitive markets (2.9) may be written as

$$\frac{w_j}{w_k} = \frac{\partial q_i/\partial v_{ij}}{\partial q_i/\partial v_{ik}}\ . \tag{2.10}$$

The interpretation of (2.10) is that the price ratio of any two variable inputs used in the production of good i must be equal to the corresponding ratio of their marginal physical products in the production of good

[1]The inequalities now become equations.

i.[1] Alternatively, the productive yield of the last dollar spent on variable input j for production of good i must be equal to that of the last dollar spent on variable input k used in production of good i.

If on the other hand the assumption that $v_{ij} > 0$, $j = (1, \ldots, s)$, is dropped and it is assumed that $v_{ij} > 0$ while $v_{ik} = 0$, $j, k \varepsilon (1, \ldots, s)$,[2] and furthermore that the strict inequality holds in the k^{th} inequality of (2.7), then (2.10) should appear in the altered form

$$\frac{\partial q_i/\partial v_{ij}}{\partial q_i/\partial v_{ik}} > \frac{w_j}{w_k} \tag{2.11}$$

or

$$\frac{\partial q_i/\partial v_{ij}}{w_j} > \frac{\partial q_i/\partial v_{ik}}{w_k} . \tag{2.12}$$

The interpretation of (2.12) is that the marginal productive yield of a dollar spent on input j in the production of good i is greater than that of a dollar spent on input k in the production of good i.[3]

[1] Where w_j is the competitive price of v_j and w_k is the competitive price of v_k.

[2] $j \neq k$.

[3] Which is of course the reason why $v_{ik} = 0$. Further results with respect to variable input usage could be imagined, however interpretations depend on values in terms of net revenues, a factor which could be quite misleading at this stage, given the nature of the decision variables of the multiproduct firm. This observation should be borne out by a glance at Chapter IV.

Before proceeding any further in this inspection of necessary conditions, perhaps it should be pointed out, especially in light of the familiarity of those conditions already considered, that the multi-product firm, even at this early stage of the game, is more than a mere collection of individual firms in that:

(a) there was an initial decision made with respect to the allocation of fixed factors among the various production processes the firm engages in;

(b) the point of production actually chosen, that is the ultimate output mix, depends on factors other than those determining output level for the single product firm regardless of the degree of competition it faces;[1]

(c) there is, under altered conditions, the possibility of switching fixed factors among alternative production processes.[2]

Returning now to the task interrupted, consider the set of conditions denoted by (2.7) and (2.8). If (V^*, Y^*) does indeed yield a cost minimum then $\lambda_i > 0$ in the sense that increases in output levels require additional variable and/or fixed factors. In particular efficient operation implies that

$$\frac{\partial C_i}{\partial q_i} \frac{\partial q_i}{\partial v_{ij}} = \lambda_i \frac{q_i}{v_{ij}} > 0 . \tag{2.13}$$

[1]This consideration, (b), will receive much attention in Chapter IV.

[2]See Chapter III and the Appendix.

Obviously, unless forced to, a firm will not employ a variable factor whose marginal physical product is ≥ 0. Note also that efficiency considerations lead the firm to increase fixed factor usage as long as such increases yield marginal products which are positive.[1] The reason for this being that fixed factors represent a sunken cost and therefore their productive possibilities will be exhausted before fresh funds are allocated for the purchase of variable factors.

Consider now a movement along the isosurface $\bar{q}_i$ in the neighborhood of (V^*, Y^*), where

$$\left(\frac{\partial q_i}{\partial y_{ik}}\right)_{y_{ij}\,\forall j \neq k} dy_{ik} + \sum_{j=1}^{s} \left(\frac{\partial q_i}{\partial v_{ij}}\right) dv_{ij} = 0 \,. \qquad (2.14)$$

Equation (2.14) indicates that the increased usage of a fixed factor, where efficient, allows the firm to reduce usage of variable factors thereby reducing the variable costs of producing the given output.[2] Therefore fixed factor usage will be intensified as long as such is feasible and efficient.

---

[1]MP = 0 = > indifference, however, it shall be assumed that the firm stops employing additional units of a fixed factor at the initial level of usage at which its marginal product disappears.

[2]All this merely says, in more familiar form, that an increase in fixed factors, wehre efficient, denotes a parametric shift in the production function yielding the same output level at lower usage of variable factors and therefore lower cost. However the above development is more in line with the direction to be taken in this chapter and the following ones.

Look now at the i^{th} set of the r sets of equations denoted by (2.8) and consider in particular the case where all around excess capacity exists, that is the instance in which $y_{ik} < \bar{y}_{ik}$, k = (1, ..., t). If in addition $y_{ik} > 0$ then

$$\lambda_i \frac{\partial q_i}{\partial y_{ik}} = 0 = \frac{\partial c_i}{\partial q_i} \frac{\partial q_i}{\partial y_{ik}} \quad i = (1, \ldots, r), \quad k = (1, \ldots, t) \qquad (2.15)$$

where $\lambda_i > 0$ implies that $\partial q_i/\partial y_{ik} = 0$, which is the conclusion mentioned above (e.g., intensity usage of y_{ik} until its marginal physical product disappears). An intuitive approach to this conclusion is offered below in Figure 1 where the good being considered is Q_1, the production function is $q_1 = q_1(v_{11}, y_{11})$ and the graphics are adopted from Krauthamer (17).

In Figure 1, OB represents the ridge line along which the marginal physical product of y_{11} vanishes; (y_{11}^*, v_{11}^*) minimizes the cost of producing Q_1 at the specified level $\bar{q}_1$ and $\bar{y}_{11} - y_{11}^*$ denotes excess capacity. It can be seen from Figure 1 that the firm will expand along OB until full capacity $(\bar{y}_{11})$ is reached. This is essentially the meaning of (2.15) except for the fact that $y_{11} = \bar{y}_{11}$ was excluded from consideration in that case.

Figure 1, in a sense, offers the implicit warning that at the output level $\bar{q}_1$ it is y_{11}^* that should appear in the production function rather than $\bar{y}_{11}$. Dano [8] is quite adamant on this very point.

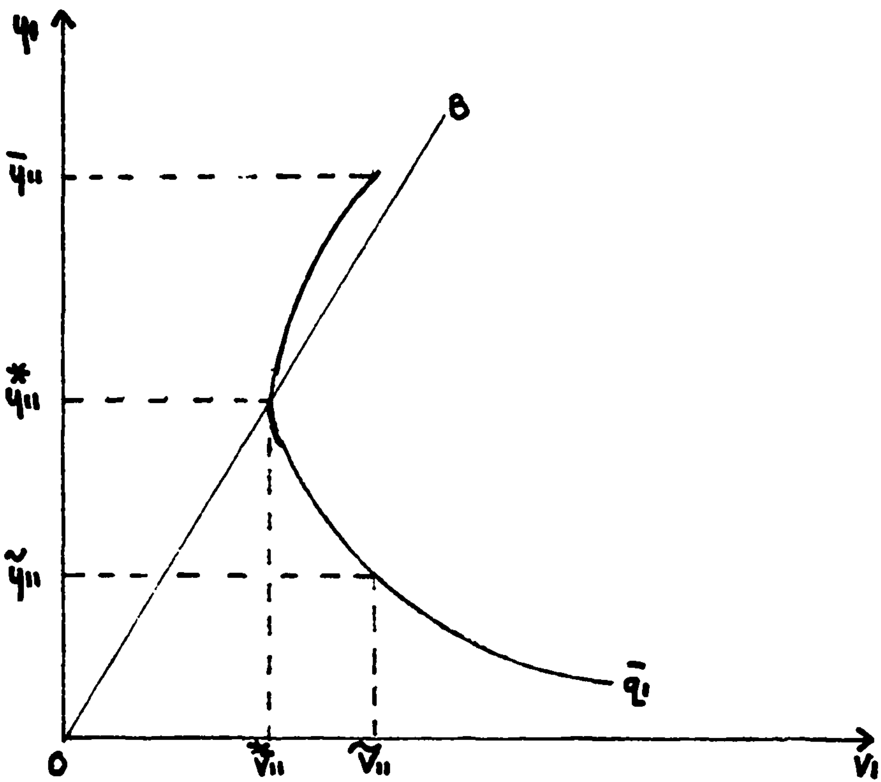

Figure 1. Ridge line

Finally, a graphical interpretation of (2.14) is given through the consideration of a movement along the isoquant, the movement emanating from ($\tilde{y}_{11}$, $\tilde{v}_{11}$) and terminating at (y^*_{11}, v^*_{11}) where the reduction in costs[1] is given by $w_1(\tilde{v}_{11} - \tilde{v}^*_{11})$ and (2.14) appears as the approximation

[1]This procedure of demonstrating the effects of increasing fixed factors on variable costs will be given a slightly more rigorous treatment in the later part of this chapter.

$$\left[\frac{\partial q_1}{\partial v_{11}} (\tilde{v}_{11} - v^*_{11}) - \frac{\partial q_1}{\partial y_{11}} (y^*_{11} - \tilde{y}_{11})\right]_{(\tilde{v}_{11},\tilde{y}_{11})} = 0 \ . \qquad (2.16)$$

The next case to be considered is that in which $y^*_{ik} = \bar{y}_{ik}$ and $u_{ik} = 0$. In this instance the constraint on fixed factor k in production of good i is barely binding, that is, a reduction in usage of that fixed factor would be detrimental in the sense that the output level $\bar{q}_i$ would not be achieved under cost minimizing conditions, while an increased usage level of fixed factor k, were such feasible, could not reduce the costs of producing the good Q_1 at the specified level. The condition for such a situation is given by picking the appropriately subscripted equation from (2.15) and appending it with $y^*_{ik} = \bar{y}_{ik}$. For all around barely binding constraints the condition is (2.15) with the additional stipulation that $y^*_{ik} = \bar{y}_{ik}$, i = (1, ..., r), k = (1, ..., t). Graphically, looking again at the production of Q_1 in the two input case the portrayal is as given in Figure 2 (below) where the interpretation warrants no further discussion.

If $y^*_{ik} = \bar{y}_{ik}$, $u_{ik} = 0$, k = (1, ..., t), and $\bar{q}_1$ is the profit maximizing level of output of Q_1 then this firm might be called a perfect capacity planner with respect to production of Q_1 [given that (2.7) is fulfilled). If $y^*_{ik} = \bar{y}_{ik}$, $u_{ik} = 0$, i = (1, ..., r), k = (1, ..., r) and $\bar{Q}$ is the profit maximizing output then the firm in question might be termed a perfect capacity planner in the sense that even if switching were feasible at a zero cost the firm would abstain from any change in

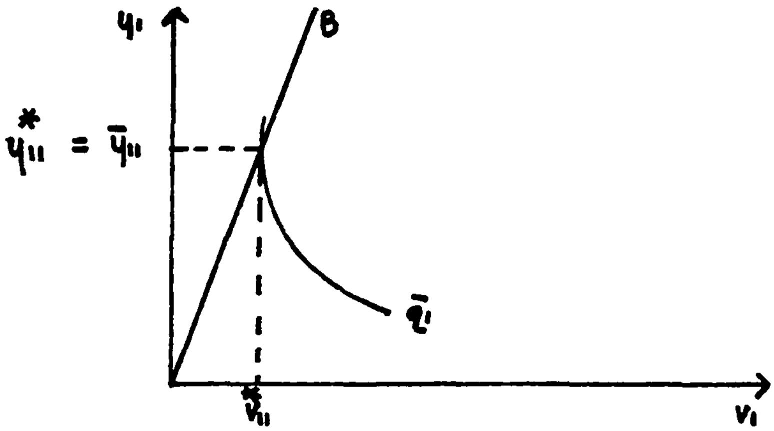

Figure 2. Barely binding constraint

its original allocation of fixed factors among the various production processes. One might then proceed a bit further along the above track and go on to define efficiency in production as the satisfaction of (2.7) and (2.8) plus the requirement that at the actual production levels chosen all the constraints on fixed factors must be barely binding.[1]

There are however many reasons why such an extension might be of somewhat limited usefulness, the first and most obvious reason being that it might seem somewhat silly to attempt to posit planning criteria in the context of a problem that excludes explicit consideration of

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[1]The barely binding criterion may, perhaps, appear to be on different footing than (2.7) and (2.8) and for that matter different than efficiency criteria as normally seen for the single product firm [(2.7); $i = 1$, $\bar{y} = y$ treated as parameter in production function]. In particular the reason for the difference is that the barely binding criterion deals with one point rather than all points on a particular expansion path, however it would seem somewhat unreasonable to talk about capacity planning at other than the production point chosen.

investment activities at the outset of the analysis. However the problem the firm faces, that is in particular the capacity it has available is of course the outcome of previous decisions and therefore the presence of excess or insufficient capacity devoted solely to production of current saleable goods and N.P.O.V.'s should indeed offer some clue as to the firm's degree of efficiency with respect to capacity planning.[1]

Another objection might be that the trouble with such an extension of efficiency criteria for the firm is that although a bit more general than previous criteria they are still partial in the same sense that Samuelson [23] notes the partiality of general equilibrium analysis.[2] Again this would be a cogent point and gratefully acknowledged. Therefore, in light of tenuousness of position the perfect capacity planning case is not posited as any sort of criterion but merely as an observation.

Continuing on again with the inspection of necessary conditions one might say that it is not at all clear that $u_{ik} > 0$ has any positive implications for planning other than as a possible signal that the firm might wish to investigate the possibility of expanding capacity in the future, however such decisions involve a myriad of considerations outside the scope of this study. In the problems to follow (Chapter III

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[1]Obviously the last statement is not in nature of a rebuttal since any real consideration of investment requires a somewhat different frame of reference than the one chosen for this analysis.

[2]In particular externalities in production have not been, nor shall they be, mentioned.

and IV) u_{ik} will be interpreted in terms of opportunity cost and net revenue, however in the present problem there is no switching which in a sense emasculates the notion of opportunity costs (that is, in the sense of how that concept will be used later on) and as noted previously, bringing marginal revenue terms into the analysis at this point would be quite misleading. One can say, however that $u_{ik} < 0$ and $u_{jk} > 0$, where $i, j \varepsilon (1, \ldots, r)$ and again the actual level of production is the point of interest, indicates an initial misallocation of fixed factor k, too much of it having been allocated to production of Q_i and not enough for Q_j.

The following is a summary[1] of possibilities already (explicitly or implicitly) examined with respect to fixed factor usage:

(a) $U^*_{ik} = 0$ and $0 \leq y^*_{ik} < \overline{y}_{ik}$, $i\varepsilon(1, \ldots, r)$, $k\varepsilon(1, \ldots, r)$,

implies excess of fixed factor k in production of Q_i;

(b) $U^*_{ik} = 0$ and $0 \leq y^*_{ik} < \overline{y}_{ik}$, $i\varepsilon(1, \ldots, r)$, $k = 1, (1, \ldots, t)$,

implies all around excess capacity in the production of Q_i

(c) $U^*_{ik} = 0$ and $0 \leq y^*_{ik} < \overline{y}_{ik}$, $i = (1, \ldots, r)$, $k = (1, \ldots, t)$,

[1]Of course, the set of all possible cases is not exhausted in the listing given. Other cases may be easily imagined but are not treated on the grounds that there is something to be said for the avoidance of tedium.

implies all around excess capacity.

(d) $U^*_{ik} = 0$ and $y^*_{ik} = \bar{y}_{ik}$, $i\varepsilon(1, \ldots, r)$, $k = (1, \ldots, t)$,

implies perfect capacity planning production of Q_i

(d) $U^*_{ik} = 0$ and $y^*_{ik} = \bar{y}_{ik}$, $i = (1, \ldots, r)$, $k = (1, \ldots, t)$,

implies perfect capacity planning.

(e) $U^*_{ik} > 0$, $U^*_{jk} = 0$, $y^*_{ik} = \bar{y}_{ik}$, $y^*_{jk} < \bar{y}_{jk}$, $i, j\varepsilon(1, \ldots, r)$,

$k\varepsilon(1, \ldots, t)$

implies a misallocation of factor k.

(f) $U^*_{ik} > 0$, $i = (1, \ldots, r)$, $k\varepsilon(1, \ldots, b)$,

implies possibility of expansion of stock of factor k in future.

At this juncture it may be of interest to consider the result brought about by the explicit introduction and consideration of those factors necessary to the usage of fixed factor (e.g., energy requirements, maintenance). The crux of the matter shall be (intuitively) that the introduction of the above consideration will cause a contraction of the relevant economic region of substitution. That is, unless those aforementioned costs of operation are negligible the firm will intensify its usage of a given fixed factor not until the marginal physical product of that factor vanishes, but rather to the point where an increase in operating costs due to capacity intensification are just equal to the accompanying reduction in variable costs engendered through such intensification. Graphically one might envision the modification of Figure 1 offered by Figure 3 (below).

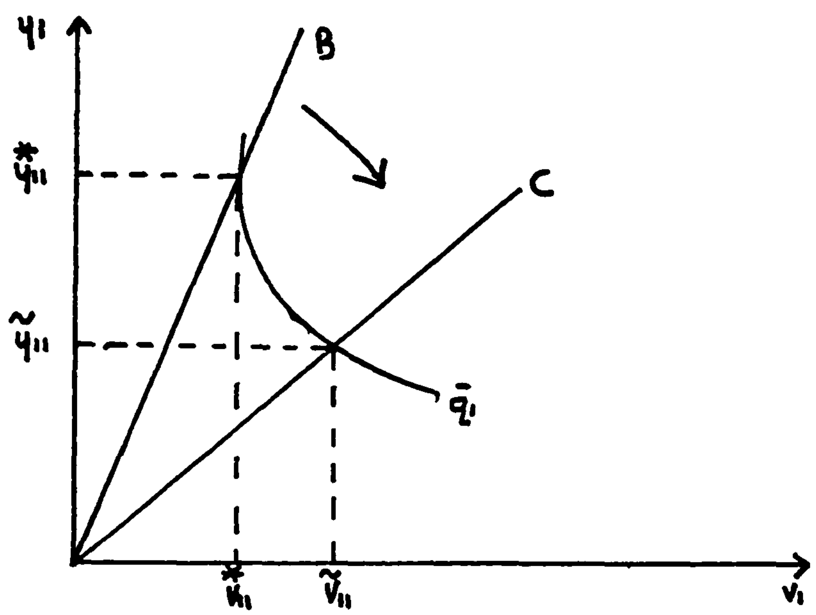

Figure 3. Introduction of operating costs

In Figure 3 positive operating costs are associated with the usgae of y_{11} and the "economic" ridge line appears to the right of the ridge line OB along which the marginal physical product of y_{11} is identically zero. Furthermore the combination of inputs which minimize the cost of production of Q_1 at the level q_1 is now pictured as $(\tilde{v}_{11}, \tilde{y}_{11})$ rather than (v^*_{11}, y^*_{11}).

For the mathematical conclusions of relevance consider the now modified cost function

$$C = C[V, Z(Y)] + F \tag{2.17}$$

$$C = (c_1, \ldots, c_i, \ldots, c_r)$$

$$Z = (z_1, \ldots, z_g) \ .$$

Z is a vector of variable inputs associated directly with the operation of fixed factors.[1] It shall be assumed that the cost minimizing combinations of these specialized variable inputs have already been determined so that the positing of a usage level y^*_{ik} immediately allows one to, in turn, posit the associated (minimized) operating costs.[2]

Except for the inclusion of Z(Y) in (2.17) the cost minimization problem is essentially the same as that given by (2.3) and (2.4). Therefore consider the modified lagrangean expression

$$L'(V, Y, \lambda, U) = C[V, Z(Y)] + \langle\lambda, \overline{Q}-Q\rangle + \langle U, G\rangle \tag{2.18}$$

---

[1]They do not appear (in this treatment) in the vector of variable factors $V = (v_1, \ldots, v_s)$.

[2]Z(Y) could also be included in the production function, however to do so eventually leads to uncalled for difficulties with respect to both manipulation of terms and assignments of marginal products.

where G is the vector of constraints on the fixed factors. If the normality condition is fulfilled at $(\tilde{V}, \tilde{Y})$ and the first order necessary conditions hold when evaluated at $(\tilde{V}, \tilde{Y})$, then a further necessary condition for $(\tilde{V}, \tilde{Y})$ to minimize C locally[1] is that

$$<\overline{V}_{qi}(\tilde{V}, \tilde{y}), \Delta i> = 0 \tag{2.19}$$

where v_{qi} is the gradient vector of the i^{th} equality constraint, and

$$\Delta i = (v_{i1}-\tilde{v}_{i1}, \ldots, v_{ij}-\tilde{v}_{ij}, \ldots, v_{is}-\tilde{v}_{is}, y_{i1}-\tilde{y}_{i1}, \ldots,$$

$$y_{ik}-\tilde{y}_{ik}, \ldots, y_{it}-\tilde{y}_{it}) .$$

Now setting i = 1 and $\partial q_i/\partial y_{ik} = 0 \; \forall \; k \neq 1$ and setting i = 1 yields

$$<(\frac{\partial q_1}{\partial v_{11}}, \ldots, \frac{\partial q_1}{\partial v_{1j}}, \ldots, \frac{\partial q_1}{\partial v_{1s}}, \frac{\partial q_1}{\partial y_{11}})(\tilde{v}, \tilde{y}) ,$$

$$(v_{11}-\tilde{v}_{11}, \ldots, v_{1j}-\tilde{v}_{1j}, \ldots, v_{1s}-\tilde{v}_{1s}, y_{11}-\tilde{y}_{11})> = 0 \tag{2.20}$$

---

[1] If $(\tilde{V}, \tilde{Y})$ is a local minimum then there exists a δ neighborhood of $(\tilde{V}, \tilde{Y})$ such that $C(\tilde{V}, \tilde{Y}) \leq C(V, Y)$ for all $(V, Y) \in NBd_\delta(\tilde{V}, \tilde{Y})$ intersection K, where $k = \{V, Y \in R^{s+t}\ \overline{q}_i - q_i = 0,\ \overline{y}_{ik} - y_{ik} \geq 0,\ V, Y \geq 0\}$ and $(\tilde{V}, \tilde{Y}) \in K$.

Actually one has a condition on the positive semi-definiteness of the hessian matrix of L that is $(V-\tilde{V}, Y-\tilde{Y}) \cdot H \cdot (V-\tilde{V}, Y-\tilde{Y}) \geq 0$, where H is the hessian of L, and it is this necessary condition that leads to satisfaction of (2.19) (below). See Appendix.

Consider now the case where $(v_{1j}-\tilde{v}_{1j}) \leq 0$, where $j = (1, \ldots, s)$ and the strict inequality holds for at least one $j\varepsilon(1, \ldots, s)$. Furthermore it is assumed that $(y_{11}-\tilde{y}_{11}) > 0$. One then has

$$\frac{\partial q_1}{\partial y_{11}} (y_{11}-\tilde{y}_{11}) = \sum_{j=1}^{s} \frac{\partial q_i}{\partial v_{1j}} (\tilde{v}_{1j}-v_{1j}) \ . \tag{2.21}$$

Without the explicit inclusion of the vector Z in the cost function one would be led to the conclusion that the total reduction in costs implies by the right hand side of (2.21) is

$$\sum_{j=1}^{s} w_j (\tilde{v}_{1j}-v_{1j}) \ .$$

However the fact is that Z has been explicitly included in (2.18) and therefore it becomes necessary to take account of the operating costs associated with the left hand side of (2.21). Those costs are given by

$$\sum_{\ell=1}^{g} \frac{\partial C_1}{\partial Z_\ell} \frac{\partial Z_\ell}{\partial y_{11}} (y_{11}-\tilde{y}_{11}) = \sum_{\ell=1}^{g} W_{Z_\ell} (Z\ell-Z\tilde{\ell}) \tag{2.22}$$

where $W_{Z\ell}$ is the competitive wage rate associated with Z_ℓ. Therefore the relevant change in costs associated with (2.21) is

$$\Delta C = \sum_{j=1}^{s} W_j(\tilde{v}_{1j}-v_{1j}) - \sum_{\ell=1}^{g} W_{Z_\ell}(Z\ell- \tilde{Z}\ell) \ . \tag{2.23}$$

Condition (2.23) yields the conclusion that the change in costs associated with the increased usage of fixed factor y_{11} will reach zero at a lower level of usage than it would had operating costs not been considered explicitly. As such this is the rationale for Figure 2 and the explanation which precedes it.

Finally, utilizing (2.23) one sees that

$$\Delta C \lesseqgtr 0 \text{ as } \left| \sum_{j=1}^{s} W_j(\tilde{v}_{ij}-v_{ij}) \right| \gtreqless \left| \sum_{\ell=1}^{g} W_{Z_\ell}(Z\ell-\tilde{Z}\ell) \right| \tag{2.24}$$

and that the firm will increase its usage of fixed k in process 1 if $\Delta C < 0$ and iff $\Delta C \leq 0$.[1]

---

[1]Of course this last result really needs no mathematics to stand upon, that is if one is minimizing costs he does not, by definition, move from a less expensive to a more expensive input combination to produce the same output. The condition (2.24) can of course be generalized to take account of any process and any factor.

CHAPTER III. CLASSICAL JOINT PRODUCTION

The second cost minimization problem to be considered deals with a multiproduct firm that operates in a productive atmosphere characterized by what Dano [8] refers to as classical joint production. The technical aspects of this type of production are also discussed by Dano's mentor Frisch [11] and by Sune Carlson [6]. One might find this type of production represented mathematically as the implicit function

$$F(q_1, \ldots, q_i, \ldots, q_r, v_1, \ldots, v_j, \ldots, v_s, y_1, \ldots, y_k, \ldots, y_t) = 0 \quad (3.1)$$

or in the alternate form

$$q_1 = q_1(q_2, \ldots, q_i, \ldots, q_r, v_1, \ldots, v_j, \ldots, v_s, y_1, \ldots, y_k, \ldots, y_t)$$

$$q_i = q_i(q_1, \ldots, q_{i+1}, \ldots, q_r, v_1, \ldots, v_j, \ldots, v_s, y_1, \ldots, y_k, \ldots, y_t)$$

$$q_r = q_r(q_1, \ldots, q_i, \ldots, q_{r-1}, v_1, \ldots, v_j, \ldots, v_s, y_1, \ldots, y_k, \ldots, y_t) \quad (3.1')$$

The scheme represented by (3.1) and (3.1') diverges from the usual only in the sense of inclusion of levels of fixed factor usage in the production function.

This type of production might be characterized by imagining the firm as engaging in one complex process in which it simultaneously turns out desired, feasible, levels of the r goods it produces. Alternatively one might think of many processes being conducted under the same roof and sharing fixed factors in the sense that switching of fixed factors between processes is not only feasible but also costless.

The function F, as it appears in (3.1), is assumed to possess continuous first and second order partial derivatives on the nonnegative orthant of R^{r+s+t}. In particular the following derivatives are of interest

$$\partial q_i/\partial v_j,\ \partial q_i/\partial yk,\ i = (1, \ldots, r),\ j = (1, \ldots, s),\ k = (1, \ldots, t)$$

$$\partial v_j/\partial y_k,\ \partial y_k/\partial v_j,\ j = (1, \ldots, s),\ k = (1, \ldots, t)$$

$$\partial q_j/\partial q_k,\ \partial y_a/\partial y_b,\ \partial v_c/\partial v_d,\ j,k \in (1, \ldots, r),\ a,b \in (1, \ldots, t),$$

$$c,d \in (1, \ldots, s)\ . \qquad (3.2)$$

If K is the feasible subset [see (3.4)] of the nonnegative orthant of R^{r+s+t} it is assumed that on some subset (which is not necessarily proper) of K

$$\frac{\partial F}{\partial v_j}, \frac{\partial F}{\partial y_k} \geq 0 \text{ and } \frac{\partial F}{\partial q_i} \leq 0, \; i = (1, \ldots, r), \; j = (1, \ldots, s),$$

$$k = (1, \ldots, t) \; . \qquad (3.2')$$

Then on this subset of K,

$$\frac{\partial q_i}{\partial v_j} = - \frac{\partial F/\partial v_j}{\partial F/\partial q_i} \geq 0$$

$$\frac{\partial q_i}{\partial y_k} = - \frac{\partial F/\partial y_k}{\partial F/\partial q_i} \geq 0 \; . \qquad (3.3)$$

The derivatives denoted by (3.3) would seem to indicate in the case where the strict inequality holds, the presence of all around substitution, however it is quite possible that in some regions derivatives such as $\partial q_i/\partial v_j$ or $\partial q_i/\partial y_k$ may be negative, since F does not rule out inefficient operation, that is, given inputs any production function reveals maximum possible outputs, however the input combination noted may well lie in an uneconomic region. Assuming away such an uneconomic region at the outset through curvature assumptions really makes discussions of excess capacity quite beside the point (at least in the context of the problem under consideration). The condition (3.2') on the other hand implies the existence of an economic region of substitution and it is assumed that in the interior of this region that some or all of the derivatives given by (3.2) are strictly positive or negative.[1]

[1] All of those derivatives may be generated through usage of the implicit function theorem and signs may be found through usage of conditions (3.2') in the economic region.

Consider now the problem

minimize C(V) (3.4)

subject to

$F(Q, V, Y) = 0$

$\overline{Q} - Q = 0$

$\overline{Y} - Y = 0$

$Q, V, Y \geq 0$

where $\overline{Q} = (\overline{q}_1, \ldots, \overline{q}_i, \ldots, \overline{q}_r)$, $q_i \geq 0$, $i = (1, \ldots, r)$.

Assume now that the $\ell + r + 1 \times r + s + t$ jacobian matrix [J] (below) is of full row rank when evaluated at $(\overline{Q}, V^*, Y^*)$.[1]

$$[J] = \begin{bmatrix} \frac{\partial F}{\partial q_1} & \cdots & \frac{\partial F}{\partial q_r} & \frac{\partial F}{\partial v_1} & \cdots & \frac{\partial F}{\partial v_s} & \frac{\partial F}{\partial y_1} & \cdots & \frac{\partial F}{\partial y_t} \\ -1 & \cdots & \frac{\partial q_1}{\partial q_r} & \frac{\partial q_1}{\partial v_1} & \cdots & \frac{\partial q_1}{\partial v_s} & \frac{\partial q_1}{\partial y_1} & \cdots & \frac{\partial q_1}{\partial y_t} \\ \vdots & -1 & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial q_r}{\partial q_1} & \cdots & -1 & \frac{\partial q_r}{\partial v_1} & \cdots & \frac{\partial q_r}{\partial v_s} & \frac{\partial q_r}{\partial y_1} & \cdots & \frac{\partial q_r}{\partial y_t} \\ \frac{\partial y_1}{\partial q_1} & \cdots & \frac{\partial y_1}{\partial q_r} & \frac{\partial y_1}{\partial v_1} & \cdots & \frac{\partial y_1}{\partial v_s} & -1 & \cdots & \frac{\partial y_1}{\partial y_t} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & -1 & \vdots \\ \frac{\partial y_\ell}{\partial q_1} & \cdots & \frac{\partial y_\ell}{\partial q_r} & \frac{\partial y_\ell}{\partial v_1} & \cdots & \frac{\partial y_\ell}{\partial v_s} & \frac{\partial y_\ell}{\partial y_1} & \cdots & -1 \end{bmatrix}_{(\overline{Q}, Y^*, V^*)} \quad (3.5)$$

---

[1]Assume also that $\ell + 1 < s + t$.

In 3.5, ℓ is the number of inequality constraints that are binding at $(\bar{Q}, Y^*, V^*)$.[1] The condition (3.7) is a normality condition and has been discussed previously in Chapter II.[2]

The lagrangean expression for the cost minimization problem given by (3.4) may be written as

$$L(Q, V, Y, \lambda, u) = C(V) + \langle \lambda, -F \rangle + \langle H, \bar{Q} - Q \rangle + \langle u, \bar{Y} - Y \rangle$$

$$Q, V, Y \geq 0, \; u \geq 0 \qquad (3.6)$$

where λ is a lagrangean multiplier, $H = (h_1, \ldots, h_i, \ldots, h_r)$ is a vector of lagrangean multipliers, $u = (u_1, \ldots, u_k, \ldots, u_t)$ is a vector of lagrangian multipliers, and $\langle \; \rangle$ denotes inner product.

If (V^*, Y^*) minimizes cost then the following conditions must hold when evaluated at (V^*, Y^*)

$$\frac{\partial C}{\partial v_1} - \lambda \frac{\partial F}{\partial v_1} \geq 0$$

$$\frac{\partial C}{\partial v_j} - \lambda \frac{\partial F}{\partial v_j} \geq 0$$

$$\frac{\partial C}{\partial v_t} - \lambda \frac{\partial F}{\partial v_t} \geq 0 \qquad (3.7)$$

---

[1]The binding inequality constraints were chosen arbitrarily to be the first ℓ where $\ell < t$.

[2]See also Appendix.

$$\lambda \frac{\partial F}{\partial q_1} + h_1 \leq 0$$

$$\lambda \frac{\partial F}{\partial q_i} + h_i \leq 0$$

$$\lambda \frac{\partial F}{\partial q_r} + h_r \leq 0 \qquad (3.8)$$

$$\lambda \frac{\partial F}{\partial y_1} + u_1 \leq 0$$

$$\lambda \frac{\partial F}{\partial y_k} + u_k \leq 0$$

$$\lambda \frac{\partial F}{\partial y_t} + u_t \leq 0 \qquad (3.9)$$

$$(\overline{Q}, V^*, Y^*) \in K = \{(Q, V, Y) \mid F = 0, \overline{Y} - Y \geq 0, \overline{Q} = Q,$$
$$V, Q, Y \geq 0, u \leq 0\} \qquad (3.10)$$

Consider now the case where $\overline{Q}, V^*, Y^* > 0$. Letting $i, j \varepsilon (1, \ldots, s)$ and dividing the i^{th} by the j^{th} equation of (3.7) yields

$$\frac{\partial C/\partial V_i}{\partial C/\partial V_j} = \frac{\partial F/\partial V_i}{\partial F/\partial V_j} = - \frac{\partial V_j}{\partial V_i} \ . \qquad (3.11)$$

Now let $z \varepsilon (1, \ldots, r)$ and divide both the i^{th} and j^{th} equations of (3.7) by the z^{th} equation of (3.8). This process leaves one with

$$- \frac{\partial C/\partial V_i}{h_z} = \frac{\partial F/\partial V_i}{\partial F/\partial q_z} = - \frac{\partial q_z}{\partial V_i} \qquad (3.12)$$

and

$$- \frac{\partial C/\partial V_j}{h_z} = \frac{\partial F/\partial V_j}{\partial F/\partial q_z} = - \frac{\partial q_z}{\partial V_j} \; . \qquad (3.13)$$

Division of (3.12) by (3.13) together with the previously mentioned assumption that the firm buys its factors in competitive markets yields

$$\frac{\partial C/\partial V_i}{\partial C/\partial V_j} = \frac{w_i}{w_j} = \frac{\partial q_z/\partial V_i}{\partial q_z/\partial V_j} \; . \qquad (3.14)$$

Utilizing (3.14) together with (3.11) one has

$$\frac{w_i}{w_j} = - \frac{\partial V_j}{\partial V_i} = \frac{\partial q_z/\partial V_i}{\partial q_z/\partial V_j} \; . \qquad (3.15)$$

Condition (3.15) should be quite familiar in the sense that it merely indicates that at $(\bar{Q}, V^*, Y^*)$ the ratio of marginal costs (in this case the constant wage rates w_i and w_j) of any two variable inputs used in production of a particular good must be equal to the corresponding ratio of marginal products.[1] The analogy with the one good two input case is quite helpful, however lest inference from that case be carried too far, one should, at this juncture, try to get at least a rough geometrical

[1]In the case where $v_i = 0$, $v_j > 0$ the equality in (3.15) must be replaced with $\geq$.

picture of what is involved in this cost minimization problem. Towards that end, first pick all the feasible input combinations that yield $\bar{q}_1$, that is, pick $(V, Y) \varepsilon k_1$ where

$$k_1 = \{(V, Y) \mid F = 0,\ y - \bar{y} \geq 0,\ \bar{q}_1 - q_1 = 0,\ q_i \neq 1 = 0,$$

$$Q, V, Y \geq 0\} .$$

Such a procedure yields an s + t - 1 dimensional level or iso-surface in the nonnegative orthant of R^{s+t} (i.e. ϕ^{s+t}). Repetition of this procedure for all of the r specified output levels yields r such iso-surfaces[1] or the set of sets $K = \{k_1, \ldots, k_r\}$.

Define now the sum of sets

$$H = \sum_{i=1}^{r} k_i$$

which is again an s + t - 1 dimensional level surface in ϕ^{s+t}. Now delete from H any nonfeasible points (points that involve some non-feasible level of a fixed input) and call the remaining set of points H'. Any input combination $(V, Y) \varepsilon H'$ will yield $\bar{Q}$.[2] The first order

---

[1]Pick all feasible input combinations that yield $\bar{q}_i$; $k_i = \{(V, Y) \mid \bar{F} = 0, \bar{Y} - Y \geq 0, \bar{q}_i - q_i = 0, q_j \neq i = 0, Q, V, \bar{Y} \geq 0\}$, where $i, j \varepsilon (1, \ldots, r)$.

[2](a) As one varies the specified output bundle $(\bar{Q})$, the iso-surfaces generated may (and most probably will) intersect, however this should not cause any problem. (b) $H' = \{(V, Y) \mid F = 0, \bar{Q} = Q, \bar{Y} - Y \geq 0, Q, V, Y \geq 0\}$.

conditions then specify rules that the point(s) picked from H' must satisfy if costs are to be minimized.

Continuing on with the inspection of first order necessary conditions, division of the k^{th} by the n^{th} equation of (3.9), where k, n ε (1, ..., t), yields

$$\frac{u_k}{u_n} = \frac{\partial F/\partial y_k}{2F/2y_n} = - \frac{\partial y_n}{2y_k} \ . \tag{3.16}$$

Redividing now the k^{th} and n^{th} equations in (3.9) by the z^{th} equation of (3.8), where z ε (1, ..., r), leaves one with

$$\frac{\partial F/\partial y_k}{\partial F/\partial q_z} = \frac{u_k}{h_z} = - \frac{\partial q_z}{\partial y_k} \tag{3.17}$$

and

$$\frac{\partial F/\partial y_n}{\partial F/\partial q_z} = \frac{u_n}{h_z} = - \frac{\partial q_z}{\partial y_k} \ . \tag{3.18}$$

Dividing (3.17) by (3.18) and using that result together with (3.16) yields

$$\frac{u_k}{u_n} = - \frac{\partial y_n}{\partial y_k} = \frac{\partial q_z/\partial y_z}{\partial q_z/\partial y_n} \ . \tag{3.19}$$

In the cases where either one or both of the relevant constraints on fixed factors are (is) slack or barely binding (3.19) is undefined or vanishes [e.g., $u_n = 0$ implies that the marginal product of fixed factor n is identically 0 in process z (in fact, in all processes) and that (3.19) is undefined]. In the case where u_k, $u_n > 0$ (3.19) merely reflects the fact that the ratio of shadow prices of two scarce fixed factors[1] is equal to the ratio of the marginal products of incremental units of the fixed factors, the relative sizes of marginal products reflecting the relative abilities to reduce costs at the margin.[2]

Assuming that both normality and first order necessary conditions are fulfilled and in addition that the hessian matrix, H, of L, evaluated at $(\overline{Q}, V^*, Y^*)$ is such that $(V - V^*, Y - Y^*) \cdot H \cdot (V - V^*, Y - Y^*) \geq 0$, one can go on to posit the second order necessary condition that

$$<\left(\frac{\partial q_1}{\partial V_1}, \ldots, \frac{\partial q_1}{\partial V_s}, \frac{\partial q_1}{\partial y_1}, \ldots, \frac{\partial q_1}{\partial y_t}\right)_{(V^*,Y^*)},$$

$$(V_1 - V_1^*, \ldots, V_s - V_s^*, y_1 - Y_1^*, \ldots, y_t - y_t^*)> = 0 \;.^3 \tag{3.20}$$

---

[1] Perhaps one should not use the term shadow prices in the above context, however, all that is meant is that u_k reflects the ability of a marginal unit of factor k to reduce costs.

[2] Many other conditions might be formed through various manipulations of the first order conditions, in particular conditions relating to allocation of fixed factors at the profit margin. However, as mentioned in Chapter II, such conclusions are more appropriately relegated to Chapter IV where multipliers are interpreted in context of additions to net revenue as well as opportunity cost.

[3] Actually one here uses (3.2) rather than (3.1) so that the constraint set is $\overline{q}_i - q_i(V_1, \ldots, V_s, y_1, \ldots, y_t) = 0$, $i = (1, \ldots, r)$. That is, rewrite the lagrangean as $Z = C(V) + \lambda(\overline{Q} - Q(V, Y)) + U(\overline{Y} - Y)$ and assume F.O.C. are both necessary and fulfilled, where $\lambda = (\lambda_1, \ldots, \lambda_r)$.

Contemplating an increase in scarce fixed factors to be utilized in production of the i^{th} good, let $dV_j = V_j - V_j^* < 0$, $j = (1, \ldots, s)$, and let $dy_k = y_k - y_k^* > 0$, $k = (1, \ldots, t)$. These last two assumptions, together with (3.20), yield the expression

$$\sum_{j=1}^{s} \frac{\partial q_i}{\partial V_j} dV_j + \sum_{k=1}^{t} \frac{\partial q_i}{\partial y_k} dy_k = 0$$

or

$$\sum_{j=1}^{s} \frac{\partial q_i}{\partial V_j} \hat{d}V_j = \sum_{k=1}^{t} \frac{\partial q_i}{\partial y_k} dy_k \tag{3.21}$$

where $\hat{d}V_j = V_j^* - V_j$.

The term on the left hand side ultimately represents a reduction in costs to the tune of

$$\sum_{j=1}^{s} w_j \hat{d}V_j$$

and therefore indicates that the value of the additional fixed factors used in production of Q_1 comes about through their cost reducing abilities, a principle previously demonstrated in Chapter II.

One can show using (3.21) and setting $dy_k = 0$ that costs rise by the amount denoted by $\Sigma w_j \hat{d}V_j$ and therefore that reallocation of scarce fixed factors between processes involves opportunity costs that can be measured in terms of dollars of variable costs. However all this is done explicitly in Chapter IV.

The production function used in this chapter will be used in Chapter IV in the sense that the cost minimization problem depicted in this section will be assumed to have taken place.

CHAPTER IV. PROFIT MAXIMIZATION

In this chapter the multi-product firm is characterized as seeking to maximize net revenue[1] over a given product line. Again interest is centered upon the economically relevant information emanating from the necessary conditions for a local solution to the problem inspected. The positing of a given product line tells one at the outset that any maximum, if achieved, is no better or worse than the product line being considered, a factor which will be completely sublimated by the very nature of the mathematical methods used to treat the profit maximum problem.

In this treatment the relevant decision variables utilized by the multiproduct firm in its attempt to maximize net revenue are prices and N.P.O.V.'s, the profit maximizing vector of decision variables being chosen from some feasible set; feasibility being limited by the limits of the productive capacity of the firm.

Nonprice offer variations are, in a sense, individual aspects of the firm's sales effort, or as Holdren [15, p. 580] puts it, "any activity of the seller which is preceptibly distinct to the buyer is potentially a distinct (nonprice offer variation)." Actually, for a discussion of the sales effort and grounds for its inclusion in the analysis

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[1]No discussion is offered below on the rationale for positing profit maximization as the objective criterion of the firm, not because the matter is trivial but rather because the subject deserves more than a cursory treatment. For discussions on alternative objective criteria see Baran and Sweezy [3], McGuire [18], and Baumol [4].

one could harken back to the work of Thorstein Veblen [27] who recognized in a crystal clear fashion that the sales effort was woven into the fabric of even the very design of a commodity. However, Veblen given his particular inclinations, would have considered the decomposition of the sales effort into distinct entities, each one itself being composed of a combination of particles, a game for the misguided. Nevertheless, such a pursuit is well within the confines of normal science as it appears in economics and paradoxical as it may seem, might well be construed as an attempt to incorporate that very notion of Veblen's (e.g., the sales effort) further into the corpus of mainstream economic theory. Bob R. Holdren [14] offers a somewhat exhaustive treatment of nonprice offer variations along modern lines and although the title of his path-breaking work would tend to indicate an exclusive interest with the theory of a multiproduct firm devoted solely to retailing, that title is misleading in the sense that the work develops a general theory applicable to any multiproduct firm. Holdren discusses not only the general concept of a nonprice offer variation as it pertains to both retailing and manufacturing firms, but also discusses N.P.O.V.'s peculiar either to the former or latter. Scitovsky [24] also discusses N.P.O.V.'s in the context of the single product firm.[1]

---

[1]The fact that Scitovsky was mentioned last (and least) in no way detracts from the brilliance of his presentation but rather merely indicates that Holdren's work is more relevant to this work than is that of Scitovsky. By more relevant, of course I only mean closer.

Another concept of interest used by Holdren [14, 15] and again by his student Gary Swenson [25] is the sales function of the multiproduct firm, a concept which takes account of interdependencies among the elements of the firm's product line. Although the sales function utilizes the information yielded by the demand surface the firm faces, the very word "sales" is an improvement in that it makes somewhat more extant in the analysis the notion that the firm has, and uses, the ability to manipulate demand. However if one wishes he may alternatively view the sales function as providing the firm with only existing demand information and the firm as utilizing that information in its profit maximizing pursuit. In this sense, if one is permitted a quip, the sales function is indeed a function for all seasons.[1]

The following notation, some of it already familiar, and some of it new, will be used throughout the remainder of the work.

$$Q = (q_1, \ldots, q_i, \ldots, q_n) , \tag{4.1}$$

q_i being the level of the i^{th} saleable good;

$$P = (p_1, \ldots, p_i, \ldots, p_n) , \tag{4.2}$$

p_i being the price level of the i^{th} saleable good;

$$V = (v_1, \ldots, v_j, \ldots, v_s) , \tag{4.3}$$

---

[1]Whether such a property is an asset or liability is of course a judgement for the reader himself to make.

v_j being the level of the j^{th} variable input;

$$\overline{Y} = (\overline{y}_1, \ldots, \overline{y}_k, \ldots, \overline{y}_t) , \tag{4.4}$$

$\overline{y}_k$ being the maximum amount of factor k available to the firm during the time period under consideration.

$$Y = (y_1, \ldots, y_k, \ldots, y_t), \; Y \leq \overline{Y} , \tag{4.5}$$

y_k being the level of the k^{th} fixed input;

$$A = (a_1, \ldots, a_w, \ldots, a_m) , \tag{4.6}$$

a_w being the level of the w^{th} N.P.O.V.;

$$Q = Q(P, A) \tag{4.7}$$

is the sales function of the firm;

$$q_i = q_i(p_1, \ldots, p_i, \ldots, p_n, a_1, \ldots, a_w, \ldots, a_m) \tag{4.8}$$

is the firms sales function for the i^{th} saleable commodity.

With respect to (4.8),

$$\frac{\partial q_i}{\partial p_i} < 0, \; \frac{\partial q_i}{\partial p_{j \neq i}} \gtrless 0, \; \frac{\partial q_i}{\partial a_w} \gtrless 0, \; w = (1, \ldots, m), \; i = (1, \ldots, n) \; . \tag{4.9}$$

That $\partial q_i(\overline{p}_1, \ldots, p_i, \ldots, \overline{p}_n, \overline{a}_1, \ldots, \overline{a}_w, \ldots, \overline{a}_m)/\partial p_i < 0$ indicates that the firm faces negatively inclined demand curves for individual products over the region of interest. However, although the actual degree of competition the firm faces with respect to markets for individual goods may differ, it shall always be the case that the firm will evaluate the efficacy of changes in decision variables in light of the effects such changes engender throughout the product line rather than with respect to effects on sales of only a single good. This need to consider the overall rather than merely individual effects of changes in decision variables is perhaps the most dramatic difference between the theory of the multi and that of the single product firm. This difference will be repeatedly emphasized by the very form of the mathematical conditions to be posited below and re-emphasized through the economic interpretation of those conditions.

That $\partial q_i/\partial p_j \neq i \gtrless 0$, merely indicates that goods may be, respectively, substitutes, independent, or complementary in sales. That $\partial q_i/\partial a_w \gtrless 0$ quite straightforwardly indicates the fact that some N.P.O.V.'s affect sales of q_i positively while others make their effect felt in the opposite direction. Obviously in a store that sells primarily cigars the attempt to stimulate pipe sales can have positive effects in the sense of increasing quantity sold of pipes; can have further positive effects from bringing in customers (pipe smokers) who might also buy some cigars; and lastly may have the negative effect of reducing sales of cigars due to the fact that to many cigar smokers the added emphasis on pipes makes the establishment perceptibly less attractive.[1] Lastly $\partial q_i/\partial a_w = 0$ would indicate either that this particular aspect of the sales effort has no effect on sales of this particular product or that the change was just not large enough to shift demand perceptibly.

Finally, the production problem the firm faces is that discussed in Chapter III. However, the results could be extended to cover the production problem in Chapter II, the major difference being the necessity of dealing with the opportunity costs terms (to be developed below) in a slightly different manner. Other relevant assumptions with respect to production and cost minimization will be introduced and discussed as the need arises.

Mathematically the profit maximization problem appears as follows: maximize

$$\Pi = \langle P, Q(P, A) \rangle - C(V(Q(A, P))) - F \tag{4.10}$$

subject to

$$\bar{Y} - Y(Q(A, P)) \geq 0, \quad P, A \geq 0 . \tag{4.11}$$

---

[1] This simple discussion is meant to merely hint at rather than exhaust possible cross effects since one could easily spend a great deal of time merely discussing cross effects in even such a simple case as a cigar store. That is, admittedly, the simplicity disappears when one realizes that merely fifty brands, each coming in about 5 to 15 sizes and each size coming in as many as 5 to 7 different colors poses grounds for considerable discussion of cross effects with respect to both prices and nonprice offer variations. Perhaps this hints at the fact that the successful small businessman may be, indeed, quite an entrepreneur.

The objective function is (4.10). The firm's profit maximization problem is to pick (A^*, P^*) from a feasible set determined by the constraint set, (4.11), such that (4.10) is maximized.

The seemingly strange or unfamiliar fashion in which the cost function and constraints are written is necessitated by the choice of (A, P) as decision variables and is explained directly below. First of all it is assumed that the firm produces all feasible outputs under cost minimizing conditions. However even this assumption is not sufficient to rule out the possibility that an expression such as V(Q(A, P)) may not be a function, that is, at (A^*, P^*) the mapping may be to a set of vectors V rather than the single vector V^*. Therefore it is assumed that any level of output, $\overline{Q}$, determines a unique V. This assumption is not really that stringent since in reality it merely amounts to the fact that the firm is aware of the input combinations it will use to produce various levels of output and that these input combinations were chosen on the basis that they could yield the desired result at the cheapest cost. The same rationale is of course used for Y(Q(A, P)).

Assume now that (A^*, P^*) is a regular point of the feasible set[1] [e.g., the binding constraint set fulfills a normality condition at (A^*, P^*)] and write the lagrangean expression L as follows:[2]

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[1]See previous chapters for rank condition of Arrow-Hurwicz-Uzawa theorem or see the Appendix.

[2]The writing out of the arguments is done for the convenience of the reader.

$$L = \sum_{L=1}^{n} p_i q_i(p_1, \ldots, p_i, \ldots, p_n, a_1, \ldots, a_w, \ldots, a_m) -$$

$$C(v_1, \ldots, v_j, \ldots, v_s) + \sum_{k=1}^{t} u_k(\bar{y}_k - y_k), \quad P, A \geq 0 \qquad (4.12)$$

where

$$U = (u_1, \ldots, u_k, \ldots, u_t) \qquad (4.13)$$

is a vector of lagrangean multipliers.

If (P^*, A^*) is to maximize Π then the following necessary conditions must hold when evaluated at

$$\frac{\partial L}{\partial P_1} = q_1 + \sum_{i=1}^{n} p_i \frac{\partial q_i}{\partial p_1} - \sum_{j=1}^{s} \sum_{i=1}^{n} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial q_i} \frac{\partial q_i}{\partial p_1} - \sum_{k=1}^{t} \sum_{i=1}^{n} U_k \frac{\partial y_k}{\partial q_i} \frac{\partial q_i}{\partial p_1} \leq 0$$

$$\frac{\partial L}{\partial P_\alpha} = q_\alpha + \sum_{i=1}^{n} p_i \frac{\partial q_i}{\partial p_\alpha} - \sum_{j=1}^{s} \sum_{i=1}^{n} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial q_i} \frac{\partial q_i}{\partial p_\alpha} - \sum_{k=1}^{t} \sum_{i=1}^{n} U_k \frac{\partial y_k}{\partial q_i} \frac{\partial q_i}{\partial p_\alpha} \leq 0$$

$$\frac{\partial L}{\partial p_n} = q_n + \sum_{i=1}^{n} p_i \frac{\partial q_i}{\partial p_n} - \sum_{j=1}^{s} \sum_{i=1}^{n} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial q_i} \frac{\partial q_i}{\partial p_n} - \sum_{k=1}^{t} \sum_{i=1}^{n} U_k \frac{\partial y_k}{\partial q_i} \frac{\partial q_i}{\partial p_n} \leq 0$$

(4.14)

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[1]Although the necessary conditions must hold for any maximum here interest will eventually be centered upon a local maximum. A point $(P^*, A^*) \varepsilon K$ is said to maximize Π locally if $\Pi(P^*, A^*) \geq \Pi(P, A)$ for all $(P, A) \varepsilon Nbd_\delta(P^*, A^*) \cap K$, where K denotes the feasible set.

Below, one of the inequalities in (4.14) is expanded so that the reader may see where the funny looking expressions came from:

$$\frac{\partial L}{\partial p_1} = q_1 + p_1 \frac{\partial q_1}{\partial p_1} + \dots + p_i \frac{\partial q_i}{\partial p_1} + \dots + p_n \frac{\partial q_n}{\partial p_1} \tag{4.15}$$

$$\begin{aligned}
&- \left(\frac{\partial C}{\partial v_1} \frac{\partial v_1}{\partial q_1} \frac{\partial q_1}{\partial p_1} + \dots + \frac{\partial C}{\partial v_1} \frac{\partial v_1}{\partial q_i} \frac{\partial q_i}{\partial p_1} + \dots + \frac{\partial C}{\partial v_1} \frac{\partial v_1}{\partial q_n} \frac{\partial q_n}{\partial p_1}\right) \\
&\quad \vdots \\
&- \left(\frac{\partial C}{\partial v_k} \frac{\partial v_k}{\partial q_1} \frac{\partial q_1}{\partial p_1} + \dots + \frac{\partial C}{\partial v_k} \frac{\partial v_k}{\partial q_i} \frac{\partial q_i}{\partial p_1} + \dots + \frac{\partial C}{\partial v_k} \frac{\partial v_k}{\partial q_n} \frac{\partial q_n}{\partial p_1}\right) \\
&\quad \vdots \\
&- \left(\frac{\partial C}{\partial v_s} \frac{\partial v_s}{\partial q_1} \frac{\partial q_1}{\partial p_1} + \dots + \frac{\partial C}{\partial v_s} \frac{\partial v_s}{\partial q_i} \frac{\partial q_i}{\partial p_1} + \dots + \frac{\partial C}{\partial v_s} \frac{\partial v_s}{\partial q_n} \frac{\partial q_n}{\partial p_1}\right)
\end{aligned} \tag{4.16}$$

$$\begin{aligned}
&-U_1\left(\frac{\partial y_1}{2q_1} \frac{\partial q_1}{2p_1} + \dots + \frac{\partial y_1}{2q_i} \frac{\partial q_i}{2p_1} + \dots + \frac{\partial y_1}{2q_n} \frac{\partial q_n}{2p_1}\right) \\
&\quad \vdots \\
&-U_k\left(\frac{\partial y_k}{\partial q_1} \frac{\partial q_1}{\partial p_1} + \dots + \frac{\partial y_k}{\partial q_i} \frac{\partial q_i}{\partial p_1} + \dots + \frac{\partial y_k}{\partial q_n} \frac{\partial q_n}{\partial p_1}\right) \\
&\quad \vdots \\
&-U_t\left(\frac{\partial y_t}{\partial q_1} \frac{\partial q_1}{\partial p_1} + \dots + \frac{\partial y_t}{\partial q_i} \frac{\partial q_i}{\partial p_1} + \dots + \frac{\partial y_t}{\partial q_n} \frac{\partial q_n}{\partial p_1}\right)
\end{aligned} \tag{4.17}$$

where

$$(4.15) = q_1 + \sum_{i=1}^{n} p_i \frac{\partial q_i}{\partial p_1} ;$$

$$(4.16) = -\sum_{j=1}^{s} \sum_{i=1}^{n} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial q_i} \frac{\partial q_i}{\partial p_1}$$

and

$$(4.17) = -\sum_{k=1}^{t} \sum_{i=1}^{n} U_k \frac{\partial y_k}{\partial q_i} \frac{\partial q_i}{\partial p_1} .$$

Further necessary conditions are

$$\frac{\partial L}{\partial a_1} = \sum_{i=1}^{n} p_i \frac{\partial q_i}{\partial a_1} - \sum_{j=1}^{s} \sum_{i=1}^{n} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial q_i} \frac{\partial q_i}{\partial a_1} - \sum_{j=1}^{s} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial a_1}$$

$$- \sum_{k=1}^{t} \sum_{i=1}^{n} U_k \frac{\partial y_k}{\partial q_i} \frac{\partial q_i}{\partial a_1} - \sum_{k=1}^{t} U_k \frac{\partial y_k}{\partial a_1} \leq 0$$

$$\vdots$$

$$\frac{\partial L}{\partial a_w}$$

$$\vdots$$

$$\frac{\partial L}{\partial a_m} = \sum_{i=1}^{n} p_i \frac{\partial q_i}{\partial a_m} - \sum_{j=1}^{s} \sum_{i=1}^{n} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial q_i} \frac{\partial q_i}{\partial a_m} - \sum_{j=1}^{s} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial a_m}$$

$$- \sum_{k=1}^{t} \sum_{i=1}^{s} U_k \frac{\partial y_k}{\partial q_i} \frac{\partial q_i}{\partial a_1} - \sum_{k=1}^{t} U_k \frac{\partial y_k}{\partial a_m} \leq 0 . \qquad (4.18)$$

Expansion of one of the above inequalities, say the first, yields

$$p_1 \frac{\partial q_1}{\partial a_1} + \ldots + p_i \frac{\partial q_i}{\partial a_1} + \ldots + p_n \frac{\partial q_n}{\partial a_1} \qquad (4.19)$$

$$-\left(\frac{\partial C}{\partial v_1}\frac{\partial v_1}{\partial q_1}\frac{\partial q_1}{\partial a_1} + \ldots + \frac{\partial C}{\partial v_1}\frac{\partial v_1}{\partial q_i}\frac{\partial q_i}{\partial a_1} + \ldots + \frac{\partial C}{\partial v_1}\frac{\partial v_1}{\partial q_n}\frac{\partial q_n}{\partial a_1}\right)$$

$$-\left(\frac{\partial C}{\partial v_j}\frac{\partial v_j}{\partial q_1}\frac{\partial q_1}{\partial a_1} + \ldots + \frac{\partial C}{\partial v_j}\frac{\partial v_j}{\partial q_i}\frac{\partial q_i}{\partial a_1} + \ldots + \frac{\partial C}{\partial v_j}\frac{\partial v_j}{\partial q_n}\frac{\partial q_n}{\partial a_1}\right)$$

$$-\left(\frac{\partial C}{\partial v_n}\frac{\partial v_n}{\partial q_1}\frac{\partial q_1}{\partial a_1} + \ldots + \frac{\partial C}{\partial v_n}\frac{\partial v_n}{\partial q_i}\frac{\partial q_i}{\partial a_1} + \ldots + \frac{\partial C}{\partial v_n}\frac{\partial v_s}{\partial q_n}\frac{\partial q_n}{\partial a_1}\right) \qquad (4.20)$$

$$-\left(\frac{\partial C}{\partial v_1}\frac{\partial v_1}{\partial a_1} + \ldots + \frac{\partial C}{\partial v_j}\frac{\partial v_j}{\partial a_1} + \ldots + \frac{\partial C}{\partial v_s}\frac{\partial v_s}{\partial a_1}\right) \qquad (4.21)$$

$$-U_1\left(\frac{\partial y_1}{\partial q_1}\frac{\partial q_1}{\partial a_1} + \ldots + \frac{\partial y_1}{\partial q_i}\frac{\partial q_i}{\partial a_1} + \ldots + \frac{\partial y_1}{\partial q_n}\frac{\partial q_n}{\partial a_1}\right)$$

$$-U_k\left(\frac{\partial y_k}{\partial q_1}\frac{\partial q_1}{\partial a_1} + \ldots + \frac{\partial y_k}{\partial q_i}\frac{\partial q_i}{\partial a_1} + \ldots + \frac{\partial y_k}{\partial q_n}\frac{\partial q_n}{\partial a_1}\right)$$

$$-U_t\left(\frac{\partial y_t}{\partial q_1}\frac{\partial q_1}{\partial a_1} + \ldots + \frac{\partial y_t}{\partial q_i}\frac{\partial q_i}{\partial a_1} + \ldots + \frac{\partial y_t}{\partial q_n}\frac{\partial q_n}{\partial a_1}\right) \qquad (4.22)$$

$$-\left(U_1\frac{\partial y_1}{\partial a_1} + \ldots + U_k\frac{\partial y_k}{\partial a_1} + \ldots + U_t\frac{\partial y_t}{\partial a_1}\right) \qquad (4.23)$$

where

$$(4.19) = \sum_{i=1}^{n} p_i \frac{\partial q_i}{\partial a_1} ;$$

$$(4.20) = -\sum_{j=1}^{s}\sum_{i=1}^{n} \frac{\partial C}{\partial v_j}\frac{\partial v_j}{\partial q_i}\frac{\partial q_i}{\partial a_1} ;$$

$$(4.21) = -\sum_{j=1}^{s} \frac{\partial C}{\partial v_j}\frac{\partial v_j}{\partial a_1} ;$$

$$(4.22) = -\sum_{k=1}^{t}\sum_{i=1}^{n} U_k \frac{\partial y_k}{\partial q_i}\frac{\partial q_i}{\partial a_1}$$

and

$$(4.23) = -\sum_{k=1}^{t} U_k \frac{\partial y_k}{\partial a_1} .$$

It is also necessary that A. P, U $\geq$ 0 and $\overline{Y} - Y \geq 0$.

To interpret the term $\partial L/\partial p_1$ consider first (4.15) which is the marginal revenue associated with an infinitesimal change in the price of good one. As the individual elements of (4.15) denote, and as has been noted previously, the marginal revenue induced by the price change is actually a compendium of effects collected throughout the product line.[1] From this term alone it should be quite clear that there is indeed a dramatic difference between the factors underlying the eventual production decisions of the multi and single product firms, that being the fact that the former is explicitly concerned with the effects that

[1]It is important to realize that a term such as $P_n(\partial q_n/\partial p_1)$ is to be evaluated at (P^*, A^*) with q_i, i = (1, ..., n-1), being held fixed at the level $q_i = q_i(P^*, A^*)$.

changes in decision variables bring about in the sales of each and every item in the product line while the latter is by definition concerned solely with the sales effect on one good.

The term denoted by (4.16) represents the marginal variable costs associated with marginal changes in production induced by the aforementioned price change. As such it is simply a compendium of marginal variable costs, again collected throughout the product line.

The interpretation of the term (4.17) is, unfortunately, a bit more involved than that of the prior two terms considered. U_k is the shadow price of fixed factor k. As such it tells one what another unit (conveniently defined) of that factor is worth to the firm in terms of an increase in net revenue. However due to the formulation of (4.10), that is, with respect to the posited decision variables of the firm, the level of usage of Y_k depends directly upon (A, P) and therefore a term such as $U_k(\partial y_k/\partial q_n)(\partial q_n/\partial p_1)$ reflects the value in terms of net revenue that would be brought about if another unit of fixed factor k were available to be allocated for use in that change in production of good n called for by the change in p_1.[1] Looking back to the cost minimization problems (in particular Chapter II) it can be seen that such an addition in net revenue comes about due to the reduction in variable costs of production.[2]

[1] Of course such a term would vanish in the case where the constraint on fixed factor k was slack or barely binding.

[2] A, perhaps, more satisfying interpretation of U_k will be presented and used in the latter portion of this chapter.

Assuming now that $P^* > 0$ [1] and $Y^* < \overline{Y}$ reduces (4.14) to

$$q_1 + \sum_{i=1}^{n} p_i \frac{\partial q_i}{\partial p_1} = \sum_{j=1}^{s} \sum_{i=1}^{n} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial q_i} \frac{\partial q_i}{\partial p_1}$$

$$q_\alpha + \sum_{i=1}^{n} p_i \frac{\partial q_i}{\partial p_\alpha} = \sum_{j=1}^{s} \sum_{i=1}^{n} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial q_i} \frac{\partial q_i}{\partial p_\alpha}$$

$$q_n + \sum_{i=1}^{n} p_i \frac{\partial q_i}{\partial p_n} = \sum_{j=1}^{s} \sum_{i=1}^{n} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial q_i} \frac{\partial q_i}{\partial p_n} \qquad (4.24)$$

The interpretation of (4.24) is quite straightforward in that the equations indicate that the marginal revenue induced by a price change must equal the marginal variable costs associated with that price change.

If the products were all independent of each other in the sense that $\partial q_i/\partial p_j \neq i = 0$, then the conditions offered by (4.24), appropriately altered, might seem to indicate that this multiproduct firm would be merely a collection of single product firms. However even in this special case the indication would be misleading due to the effects (to be considered below) of the N.P.O.V.'s which make themselves felt

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[1] Indeed hardly an unlikely situation since consideration is being centered upon saleable output prices only, however, for the purist, one can assume that as the price of a good becomes arbitrarily small, the demand becomes arbitrarily large and therefore prices are bound away from zero due to the constraints on the firms productive capacity. The alert reader has probably surmised that $Y(Q(P^*, A^*)) = \hat{Y}$ such that some $\hat{y}_k > \overline{y}_k$ implies that $P^*, A^*)$ is not feasible.

throughout the product line, as well as the inclusion of decisions with respect to the allocation of scarce factors between alternative uses (again, to be considered below).

It is also interesting to note that if the firm under consideration were to produce only Q_1, then the relevant marginal revenue term with respect to a change in p_1 would be $q_1 + p_1(\partial a_1/\partial p_1)$ and that

$$q_1 + p_1 \frac{\partial q_1}{\partial p_1} \gtrless q_1 + p_1 \frac{\partial q_1}{\partial p_1} \qquad (4.25)$$

multiproduct firm   single product firm

and in fact, in the case of the multiproduct firm it may be that

$$q_1 + p_1 \frac{\partial q_1}{\partial p_1} - \sum_{j=1}^{s} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial q_1} \frac{\partial q_1}{\partial p_1} < 0 ,$$

which again merely emphasizes that notion that the multiproduct firm is interest in overall reverberations due to adjustment of decision variables rather than just individual effects.[1]

Now letting a, b ε (1, ..., n) and dividing the a^{th} by the b^{th} equation of (4.24) yield

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[1]Surely the case of loss leaders is not unbeknownst to the reader, and although loss leaders are usually associated with retailing firms they are nevertheless relevant to producers (both wholesalers and retailers).

$$\frac{q_a + \sum_{i=1}^{n} p_i \frac{\partial q_i}{\partial p_a}}{q_b + \sum_{i=1}^{n} p_i \frac{\partial q_i}{\partial p_b}} = \frac{\sum_{j=1}^{s} \sum_{i=1}^{n} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial q_i} \frac{\partial q_i}{\partial p_a}}{\sum_{j=1}^{s} \sum_{i=1}^{n} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial q_i} \frac{\partial q_i}{\partial p_b}} . \qquad (4.26)$$

It is of course the individual terms of (4.26) in which any novelty inherent in these conditions must lie since other than that one merely has a ratio of marginal gains equal to a corresponding ratio of marginal costs.

Moving on to the second set of necessary conditions and making the additional assumption that $A^* > 0$ one has the following set of equations

$$\sum_{i=1}^{n} p_i \frac{\partial q_i}{\partial a_1} = \sum_{j=1}^{s} \sum_{i=1}^{n} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial q_i} \frac{\partial q_i}{\partial a_1} + \sum_{j=1}^{s} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial a_1}$$

$$\sum_{i=1}^{n} p_i \frac{\partial q_i}{\partial a_w} = \sum_{j=1}^{s} \sum_{i=1}^{n} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial q_i} \frac{\partial q_i}{\partial a_w} + \sum_{j=1}^{s} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial a_w}$$

$$\sum_{i=1}^{n} p_i \frac{\partial q_i}{\partial a_m} = \sum_{j=1}^{s} \sum_{i=1}^{n} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial q_i} \frac{\partial q_i}{\partial a_m} + \sum_{j=1}^{s} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial a_m} . \qquad (4.27)$$

The equalities denoted by (4.27) denote that at (A^*, P^*) the gains engendered from a marginal change in a_w, $w \in (1, \ldots, m)$, must be equal to the costs ultimately assignable to that variation. In particular those costs are the marginal variable costs of producing induced changes in output as well as those emanating from the use of variable inputs to produce the posited variation in a_w.

Now letting w, z ε (1, ..., m) and dividing the w^{th} by the z^{th} equation of (4.27) yields

$$\frac{\sum_{i=1}^{n} p_i \frac{\partial q_i}{\partial a_w}}{\sum_{i=1}^{n} p_i \frac{\partial q_i}{\partial a_z}} = \frac{\sum_{j=1}^{s} \sum_{i=1}^{n} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial q_i} \frac{\partial q_i}{\partial a_w} + \sum_{j\ 1}^{s} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial a_w}}{\sum_{j=1}^{s} \sum_{i=1}^{n} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial q_i} \frac{\partial q_i}{\partial a_z} + \sum_{s=1}^{s} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial a_w}} \qquad (4.28)$$

The interpretation of (4.28) is analogous to that of (4.26). If, on the other hand, $a_z = 0$, $a_w > 0$ and the strict inequality holds in the z^{th} inequality of (4.18) then (4.28) would be modified to read

$$\frac{\sum_{i=1}^{n} p_i \frac{\partial q_i}{\partial a_w}}{\sum_{i=1}^{n} p_i \frac{\partial q_i}{\partial a_z}} > \frac{\sum_{j=1}^{s} \sum_{i=1}^{n} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial q_i} \frac{\partial q_i}{\partial a_w} + \sum_{j=1}^{s} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial a_w}}{\sum_{j=1}^{s} \sum_{i=1}^{n} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial q_i} \frac{\partial q_i}{\partial a_z} + \sum_{j=1}^{s} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial a_z}} . \qquad (4.29)$$

Actually the information of interest is contained in the z^{th} expression of (4.18) (now a strict inequality) which when interpreted merely says that the costs of increasing a_z from 0 to a possible level are greater than the gains associated with such a change, that is

$$\sum_{i=1}^{n} p_i \frac{\partial q_i}{\partial a_z} < \sum_{j=1}^{s} \sum_{i=1}^{n} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial q_i} \frac{\partial q_i}{\partial a_z} + \sum_{j=1}^{s} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial a_z} . \qquad (4.30)$$

Conditions (4.26), (4.28), and (4.29) together might be termed the first order necessary conditions for the case of all around excess capacity.

Look now at $\partial L/\partial p_1$ in the case where U > 0, that is the constraints on fixed factors are more than barely binding. The expression is reproduced directly below as (4.31).

$$q_1 + \sum_{i=1}^{n} p_i \frac{\partial q_i}{\partial p_1} = \sum_{j=1}^{s} \sum_{i-1}^{n} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial q_i} \frac{\partial q_i}{\partial p_1} + \sum_{k=1}^{t} \sum_{i=1}^{n} U_k \frac{\partial y_k}{\partial q_i} \frac{\partial q_i}{\partial p_1} \tag{4.31}$$

At first glance (4.31) does not look very familiar in the sense that the majority of expressions examined prior to this point, although composed of what one grounded in contemporary textbook price theory might term backwards from normal derivatives, seemed to, at the very least, strike somewhat familiar chords. The expression of interest, however, does not seem to strike such chords. The problem is that one does not usually see terms involving fixed factors explicitly in first order conditions although such was indeed the case in the cost minimization problems done in Chapters II and III. Ample precedence for such a procedure is offered in Pfouts [20], Benavie [5], Ferguson [10], and Naylor [19] among others although their treatments differed in several respects from the ones being offered in this work.

In the cost minimization problems[1] it was shown that the increased usage of fixed factors (when efficient) reduced the costs of producing a given product at a given level. This of course is nothing novel in the sense that it merely represents a parametric shift in the production function which increase marginal products of the variable factors thereby ultimately reducing the variable costs of maintaining a given level of output. However in the case that the constraint on a particular fixed factor is slack, that fixed factor has been used to the point where a further increase in its level of usage would be detrimental to the posited goal of the firm. Therefore an additional unit of that fixed factor is of no immediate value to the firm and the shadow price assigned to that factor is identically zero. Alternatively one might say that the presence of excess capacity with respect to fixed factor k indicates that the price the firm must pay to obtain another unit of that factor is zero. But consider now the case where fixed factor k is scarce in the sense that $y_k^* = \bar{y}_k$ and $U_k > 0$. Now the price that the firm must pay to use another unit of this factor in the production of a particular output is no longer zero in the sense that such a usage involves a reduced level of output in some other productive use where the factor could make a positive contribution. A marginal change in a decision variable, then, causes a reallocation of scarce capacity; such reallocation involving an opportunity cost no less real than any other cost and measured in dollars necessary to purchase variable

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[1]See in particular the development using second order necessary conditions in the latter portion of Chapter II.

factors necessary to compensate for capacity reduction in areas of alternative use for the scarce factor.

Consider as an approximation for that aforementioned opportunity cost with respect to a change in a decision variable, say p_1,

$$\sum_{k=1}^{t} \sum_{i=1}^{n} U_k \frac{\partial y_k}{\partial q_i} \frac{\partial q_i}{\partial p_1} dp_1 = \sum_{k=1}^{t} \sum_{i-1}^{n} U_k \frac{\partial y_k}{\partial q_i} dq_i = \sum_{k-1}^{t} U_k dy_k \qquad (4.32)$$

where (4.32) is to be evaluated at (A^*, P^*, U^*) and $dp_1 = p_1 - p_1^*$. Since U_k is the value in terms of net revenue of an additional unit of y_k, (4.32) is therefore a weighted sum of values and constitutes a sum of costs to the firm in the sense that specific amounts of fixed factors have been diverted from alternative uses. To see the costs of diversion consider

$$(q_i + \sum_{i=1}^{n} p_i \frac{\partial q_i}{\partial p_1}) dp_1 = \sum_{j=1}^{s} \frac{\partial C}{\partial v_j} dv_j + \sum_{k=1}^{t} U_k dy_k \qquad (4.33)$$

where

$$\sum_{j=1}^{s} \sum_{i=1}^{n} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial q_i} \frac{\partial q_i}{\partial p_1} dp_1 = \sum_{j=1}^{s} \sum_{i-1}^{n} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial q_i} dq_i = \sum_{j=1}^{s} \frac{\partial C}{\partial v_j} dv_j$$

$$\sum_{k=1}^{t} \sum_{i=1}^{n} U_k \frac{\partial y_k}{\partial q_i} \frac{\partial q_i}{\partial p_1} dp_1 = \sum_{k=1}^{t} \sum_{i=1}^{n} U_k \frac{\partial y_k}{\partial q_i} dq_i = \sum_{k=1}^{t} U_k dy_k \, .$$

Assuming both sums on the right hand side of (4.33) are positive set $dy_k = 0$ causing dv_j to rise to $\hat{d}v_j = dv_j + \Delta v_j$ in turn causing variable costs to rise by

$$\sum_{j=1}^{s} w_j(\Delta v_j)$$

which is a dollar sum exactly equal to

$$\sum_{k=1}^{t} U_k dy_k \ .$$

This simple procedure should demonstrate that the diversion of scarce fixed factors from alternative uses constitutes a cost as real as any other.[1]

Reconsider now (4.33) in the case where dy_k is again set equal to zero but this time dv_j is held constant, that is $\Delta v_j = 0$. In this instance the term on the left hand side of (4.33) must fall[2] by an equal amount, that is, the marginal revenue (approximation to) must fall by the amount

$$\sum_{k=1}^{t} U_k dy_k \ .$$

This (somewhat rough) conclusion together with the previous one should serve to intuitively demonstrate that the value of scarce fixed factors

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[1] All this will be demonstrated below in a slightly more rigorous fashion through the usage of second order necessary conditions.

[2] The assumptions of the case directly preceding this one are being maintained, however in the more general case one would replace the word fall by the word change.

is due to a reduction in direct variable costs and may be (in principle) measured by observing the degree to which marginal revenue falls when the usage of the aforementioned factors is restricted.

In light of the discussion directly above it becomes evident that the conditions (4.26) are a special case which hold only in the instance that U = 0, and that the general case for $U \geq 0$ is given by

$$\frac{q_a + \sum_{i=1}^{n} p_i \frac{\partial q_i}{\partial p_a}}{q_b + \sum_{i=1}^{n} p_i \frac{\partial q_i}{\partial p_b}} = \frac{\sum_{j=1}^{s} \sum_{i=1}^{n} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial q_i} \frac{\partial q_i}{\partial p_a} + \sum_{k=1}^{t} \sum_{i=1}^{n} U_k \frac{\partial y_k}{\partial q_i} \frac{\partial q_i}{\partial p_a}}{\sum_{j=1}^{s} \sum_{i=1}^{n} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial q_i} \frac{\partial q_i}{\partial p_b} + \sum_{k=1}^{t} \sum_{i=1}^{n} U_k \frac{\partial y_k}{\partial q_i} \frac{\partial q_i}{\partial p_b}} \tag{4.34}$$

Looking now at $\partial L/\partial a_1$ in the case where A > 0, $U \geq 0$ one sees

$$\sum_{i=1}^{n} p_i \frac{\partial q_i}{\partial a_1} = \sum_{j=1}^{s} \sum_{i=1}^{n} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial q_i} \frac{\partial q_i}{\partial a_1} + \sum_{j=1}^{s} \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial a_1}$$

$$+ \sum_{k=1}^{t} \sum_{i=1}^{n} U_k \frac{\partial y_k}{\partial q_i} \frac{\partial q_i}{\partial a_1} + \sum_{k=1}^{t} U_k \frac{\partial y_k}{\partial a_1} \, . \tag{4.35}$$

The main point of interest in (4.35) is that the opportunity costs associated with the variation in a_1 accrue not only through the induced changes in output but also through the production of the marginal change in a_1 itself. The appropriate generalizations of (4.28) and (4.29) are accomplished through inclusion of opportunity cost terms.

One could, of course, in the case where A > 0, $U \geq 0$, add the further condition

$$\frac{q_a + \sum_{i=1}^{n} p_i \frac{\partial q_i}{\partial p_a}}{\sum_{i=1}^{n} p_i \frac{\partial q_i}{\partial a_w}}$$

$$= \frac{\sum \sum \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial q_i} \frac{\partial q_i}{\partial p_a} + \sum \sum U_k \frac{\partial y_k}{\partial q_i} \frac{\partial q_i}{\partial p_a}}{\sum \sum \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial q_i} \frac{\partial q_i}{\partial a_w} + \sum \frac{\partial C}{\partial v_j} \frac{\partial v_j}{\partial a_w} + \sum \sum U_k \frac{\partial y_k}{\partial q_i} \frac{\partial q_i}{\partial a_w} + \sum U_k \frac{\partial y_k}{\partial a_w}} \tag{4.36}$$

This expression, (4.36), is again merely a ratio of gains set equal to a ratio of costs, but it does however reflect the fact the changes in prices are, in terms of direct costs as well as opportunity costs, cheaper to accomplish than changes in N.P.O.V.'s. That is not to say that in actuality the costs of gaining information and expertise in the setting of prices is nil, but rather that the price change itself is not a produced item while the variation in an N.P.O.V. is.[1]

If the rank condition is fulfilled and (A^*, P^*) maximizes net revenue locally, then one has

$$(A - A^*, P - P^*) \cdot G \cdot (A - A^*, P - P^*) \leq 0$$

where G is the hessian matrix of L and is to be evaluated at (A^*, P^*, U^*).

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[1]In the case $a_w = 0$, the = is replaced by $\geq$.

The above condition yields the second order necessary condition

$$<\overline{\nabla}Y_{(A^*,P^*)}, \Delta> \quad (4.37)$$

where $\Delta = (A - A^*, P - P^*)$. Now, setting $A - A^* = 0$ and $y_k \neq 1$ constant yields

$$<(\frac{\partial y_1}{\partial p_1}, \ldots, \frac{\partial y_1}{\partial p_i}, \ldots, \frac{\partial y_1}{\partial p_n})_{(A^*,P^*)},$$

$$(p_1 - p_1^*, \ldots, p_i - p_i^*, \ldots, p_n - p_n^*)> = 0 . \quad (4.38)$$

Multiplication of (4.38) by U_1 yields

$$U_1 \sum_{i=1}^{n} \frac{\partial y_1}{\partial p_i} dp_i \quad (4.39)$$

where $dp_i = p_i - p_i^*$. Noting that

$$\frac{\partial y_1}{\partial p_a} dp_a = \sum_{i=1}^{n} \frac{\partial y_1}{\partial q_i} \frac{\partial q_i}{\partial p_a} dp_a = \sum_{i-1}^{n} \frac{\partial y_1}{\partial q_i} dq_i ,$$

one can write, in the case where $dp_i \neq a = 0$, that

$$U_1 \sum_{i=1}^{n} \frac{\partial y_1}{\partial p_i} dp_i = U_1 \sum_{i=1}^{n} \frac{\partial y_1}{\partial q_i} \frac{\partial q_i}{\partial p_a} dp_a = U_1 \sum_{i=1}^{n} \frac{\partial y_1}{\partial q_i} dq_i . \quad (4.40)$$

Now arbitrarily setting

$dq_i > 0, i = (1, \ldots, d)$

$dq_i < 0, i = (d + 1, \ldots, n)$,

yields

$$U_1 \sum_{i=1}^{d} \frac{\partial y_1}{\partial q_i} dq_i = U_1 \sum_{i=d+1}^{n} \frac{\partial y_1}{\partial q_i} \hat{d}q_i \ .^1 \tag{4.41}$$

The left hand side of (4.41) measures the increase in net revenue due to the increased usage of y_1 in processes one through d while the right hand side measures the decrease in net revenue due to the removal of y_1 from processes d + 1 through n. The fact that the two sides of (4.41) are equal indicates that at (A^*, P^*) factor one is allocated in a fashion such that a reallocation (locally) could not increase profits. Alternatively, the losses in net revenue accruing due to an induced withdrawal of factor one must be just offset by the gains in net revenue which accure through its increased usage in alternative processes.

Now still holding $dp_i \neq a = 0$ and letting all fixed factors vary, yields the set of equalities given in (4.42) (below).

$$U_1 \sum_{i=1}^{d} \frac{\partial y_1}{\partial q_i} dq_i = U_1 \sum_{i=d+1}^{n} \frac{\partial y_1}{\partial q_i} \hat{d}q_i$$

---

[1] Where $\hat{d}q_i = q_1^* - q_i$.

$$U_k \sum_{i=1}^{d} \frac{\partial y_k}{\partial q_i} dq_i = U_k \sum_{i=d+1}^{n} \frac{\partial y_k}{\partial q_i} \hat{d}q_i$$

$$U_t \sum_{i=1}^{d} \frac{\partial y_t}{\partial q_i} dq_i = U_t \sum_{i=d+1}^{n} \frac{\partial y_k}{\partial q_i} \hat{d}q_i \ . \qquad (4.42)$$

Summation of the left hand and the right sides of (4.42) leaves one with the equality

$$\sum_{k=1}^{t} \sum_{i=1}^{d} U_k \frac{\partial y_k}{\partial q_i} dq_i = \sum_{k=1}^{t} \sum_{i=d+1}^{n} U_k \frac{\partial y_k}{\partial a_1} \hat{d}q_i \ . \qquad (4.43)$$

The equality (4.43) indicates that at (A^*, P^*) the net revenue created through an induced reallocation of fixed factors will be exactly equal to reduction in net revenue engendered by the removal of those fixed factors from alternative uses.

Going back now for a moment to (4.37) and hold $dp_i \neq a, b$ constant one gets the set of equations

$$\frac{\partial y_1}{\partial p_a} dp_a + \frac{\partial y_1}{\partial p_b} dp_b = 0$$

$$\frac{\partial y_k}{\partial p_a} dp_a + \frac{\partial y_k}{\partial p_b} dp_b = 0$$

$$\frac{\partial y_t}{\partial p_a} dp_a + \frac{\partial y_t}{\partial p_b} dp_b = 0 \ . \qquad (4.44)$$

Multiplying each of the t equation of (4.44) by the appropriate lagrangean multiplier (multiply the k^{th} equation by U_k^*) together with summation of the equations yields

$$\sum_{k=1}^{t} U_k \frac{\partial y_k}{\partial p_a} dp_a = \sum_{k=1}^{t} U_k \frac{\partial y_k}{\partial p_b} dp_b \; .^{1} \qquad (4.45)$$

At the profit maximum an infinitesimal change in any price will have the same effect. If (A^*, P^*) is a unique local maximum one could amend the interpretation to read that at the profit maximum infinitesimal change in any price will reduce net revenue by the same amount. Of course the above analysis can be extended to include N.P.O.V.'s,[2] however, as noted earlier, there is something to be said for the avoidance of tedium.

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[1] Where $\hat{d}p_b = -dp_b$.

[2] For instance replace the r.h.s. of (4.45) with $\sum_{k=1}^{t} U_k \frac{\partial y_k}{\partial a_w} da_w$.

CHAPTER V. CONCLUSION

As indicated in the introduction, and as the reader must be aware of by this point, this work is not of the nature of an overall complete one in the sense that many important problems concerning the theory of the multiproduct have not been treated; some in fact have not even been mentioned. However, this work is merely of a part of what is to be (hopefully) ongoing research on the theory of the multiproduct firm.

It is hoped, however, that what has been presented has at least amply stressed the nature of the multiproduct firm's short run profit maximization decisions as well as the increased role of importance fixed factors assume in both the profit maximization and cost minimization problems. Admittedly the nature of the decision variables chosen for the profit maximization problem lead to some unfamiliar and perhaps what might be termed unwieldly expressions. On this last score three points might be made, the first one being that, of course, any way of doing things different from that way which has been in the past continually stressed and practiced would seem unfamiliar and perhaps unwieldly. The second point is that symmetry, as appealing as it is, is certainly not the end of theorizing. Thirdly, the choice of decision variables was in no way based on the hopes of product differentiation but rather upon the notion that those control variables are indeed the ones of interest to the multiproduct firm.

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APPENDIX. NONLINEAR PROGRAMMING

This Appendix treats a nonlinear programming problem which is similar in form to those problems treated throughout the preceding chapters. The treatment of this problem (to be given below) is designed to parallel that used in the above mentioned problems.

Some more general treatments particularly useful to economists are Hadley [12], Intrilligator [16], Benavie [5], and Tackayama [26]. A recent treatise by Hestenes [13], a mathematician noted for his work in optimal control theory, treats nonlinear programming via vector space methods. Of course this brief list in no way exhausts the larger set of works available on the subject.

Assume that X is an open subset of R^n and the problem is to maximize (minimize)

$$f(x_1, \ldots, x_i, \ldots, x_n), \; x = (x_1, \ldots, x_n) \in X$$

subject to

$$g_j(x_1, \ldots, x_i, \ldots, x_n) \geq 0, \; j = 1, \ldots, m$$

$$h_k(x_1, \ldots, x_i, \ldots, x_n) = 0, \; k = 1, \ldots, \ell$$

$$x_i \geq 0, \; i = 1, \ldots, n \qquad \text{(A.1)}$$

where f, g_j and h_k, j = (1, ..., m), k = (1, ..., ℓ). are assumed to be real valued and twice differentiable on X.[1]

Assume that $x^* \varepsilon K$ is such that $f(x^*) \geq f(x) \quad x \varepsilon Nbd_\delta(x^*) \cap K$ where $K = \{x | g_j \geq 0, h_k = 0, x \geq 0\}$ and $x^* \varepsilon K$. Furthermore, assume that the following jacobian matrix J is of full row rank,

$$(J) = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} \cdots \frac{\partial h_1}{\partial x_i} \cdots \frac{\partial h_1}{\partial x_n} \\ \vdots \\ \frac{\partial h_k}{\partial x_1} \cdots \frac{\partial h_k}{\partial x_i} \cdots \frac{\partial h_k}{\partial x_n} \\ \vdots \\ \frac{\partial h_\ell}{\partial x_1} \cdots \frac{\partial h_\ell}{\partial x_i} \cdots \frac{\partial h_\ell}{\partial x_n} \\ \vdots \\ \frac{\partial g_1}{\partial x_1} \cdots \frac{\partial g_1}{\partial x_i} \cdots \frac{\partial g_1}{\partial x_n} \\ \vdots \\ \frac{\partial g_d}{\partial x_1} \cdots \frac{\partial g_d}{\partial x_i} \cdots \frac{\partial g_d}{\partial x_n} \end{bmatrix}_{d + \ell \times n} (X^*) \qquad \text{(A.2)}$$

where d denotes the number of effective inequality constraints at X^*, $d < m$, and $\ell + d < n$.

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[1]Actually if interest is merely of a local nature one need make such differentiability assumptions only on a neighborhood of the solution.

Now from the lagrangean expression

$$L(X, \lambda, U) = f(x_1, \ldots, x_i, \ldots, x_n)$$

$$+ \sum_{k=1}^{\ell} \lambda_k h_k(x_1, \ldots, x_i, \ldots, x_n)$$

$$+ \sum_{j=1}^{m} U_j g_j(x_1, \ldots, x_i, \ldots, x_n) \qquad \text{(A.3)where}$$

where

$$\lambda = (\lambda_1, \ldots, \lambda_k, \ldots, \lambda_\ell)$$

and

$$U = (U_1, \ldots, U_j, \ldots, U_m)$$

and

$$x_i \geq 0, \; i = (1, \ldots, n); \; U_j \geq 0, \; j = (1, \ldots, m) \; .$$

If X^* maximizes[1] f locally subject to the constraints given in (A.1) then the following conditions (first order necessary conditions) must hold at X^*

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[1]For minimization reverse the inequalities in (A.3) and (A.4).

$$\frac{\partial f}{\partial x_1} + \sum_{k=1}^{\ell} \lambda_k \frac{\partial h_k}{\partial x_1} + \sum_{j=1}^{m} U_j \frac{\partial g_j}{\partial x_1} \leq 0$$

$$\frac{\partial f}{\partial x_i} + \sum_{k=1}^{\ell} \lambda_k \frac{\partial h_k}{\partial x_i} + \sum_{j=1}^{m} U_j \frac{\partial g_j}{\partial x_i} \leq 0$$

$$\frac{\partial f}{\partial x_n} + \sum_{k=1}^{\ell} \lambda_k \frac{\partial h_k}{\partial x_n} + \sum_{j=1}^{m} U_j \frac{\partial g_j}{\partial x_n} \leq 0 \ . \qquad \text{(A.3)}$$

If X^* is a local maximum and (A.2) is satisfied then

$$(X - X^*) \cdot H \cdot (X - X^*) \leq 0 \qquad \text{(A.4)}$$

where H is the hessian matrix of L.

Condition (A.4) satisfies

$$<(\frac{\partial h_k}{\partial x_1}, \ \ldots, \ \frac{\partial h_k}{\partial x_i}, \ \ldots, \ \frac{\partial h_k}{\partial x_n}),$$

$$(x_1 - x_1^*, \ \ldots, \ x_i - x_i^*, \ \ldots, \ x_n - x_n^*)> = 0$$

where $k = (1, \ldots, \ell)$, and

$$<(\frac{\partial g_j}{\partial x_1}, \ \ldots, \ \frac{\partial g_j}{\partial x_i}, \ \ldots, \ \frac{\partial g_j}{\partial x_n}),$$

$$(x_1 - x_1^*, \ \ldots, \ x_i - x_i^*, \ \ldots, \ x_n - x_n^*)> = 0$$

where $j \in J = (1, \ldots, d)$, and J is the set of inequality constraints that are binding at X^*.